# One Factor Experiments


- Computation of Effects
- Estimating Experimental Errors
- Allocation of Variation
- ANOVA Table and F-Test
- Visual Diagnostic Tests
- Confidence Intervals For Effects
- Unequal Sample Sizes

# One Factor Experiments

- Used to compare alternatives of a single categorical variable.

$$y_{ij} = \mu + \alpha_j + e_{ij}$$

For example, several processors, several caching schemes

$$\begin{aligned}
\text{r} &= \text{Number of replications} \\
y_{ij} &= \text{ith response with jth alternative} \\
\mu &= \text{mean response} \\
\alpha_j &= \text{Effect of alternative j} \\
e_{ij} &= \text{Error term}
\end{aligned}$$

$$\sum \alpha_j = 0$$

# Computation of Effects

$$\sum_{i=1}^{r}\sum_{j=1}^{a} y_{ij} = ar\mu + r\sum_{j=1}^{a}\alpha_j + \sum_{i=1}^{r}\sum_{j=1}^{a} e_{ij}$$

$$= ar\mu + 0 + 0$$

$$\mu = \frac{1}{ar}\sum_{i=1}^{r}\sum_{j=1}^{a} y_{ij} = \bar{y}_{..}$$

# Computation of Effects (Cont)

$$
\begin{aligned}
\bar{y}_{.j} &= \frac{1}{r}\sum_{i=1}^{r} y_{ij} \\
&= \frac{1}{r}\sum_{i=1}^{r}(\mu + \alpha_j + e_{ij}) \\
&= \frac{1}{r}\left(r\mu + r\alpha_j + \sum_{i=1}^{r} e_{ij}\right) \\
&= \mu + \alpha_j + 0
\end{aligned}
$$

$$\alpha_j = \bar{y}_{.j} - \mu = \bar{y}_{.j} - \bar{y}_{..}$$

# Example 20.1: Code Size Comparison

| R | V | Z |
|---|---|---|
| 144 | 101 | 130 |
| 120 | 144 | 180 |
| 176 | 211 | 141 |
| 288 | 288 | 374 |
| 144 | 72 | 302 |

- Entries in a row are unrelated.
  (Otherwise, need a two factor analysis.)

# Example 20.1 Code Size (Cont)

| | R | V | Z | |
|---|---|---|---|---|
| | 144 | 101 | 130 | |
| | 120 | 144 | 180 | |
| | 176 | 211 | 141 | |
| | 288 | 288 | 374 | |
| | 144 | 72 | 302 | |
| Col Sum | $\Sigma\ y_{.1}$ =872 | $\Sigma\ y_{.2}$ =816 | $\Sigma\ y_{.3}$ =1127 | $\Sigma\ y_{..} = 2815$ |
| Col Mean | $\bar{y}_{.1}$ =174.4 | $\bar{y}_{.2}$ =163.2 | $\bar{y}_{.3}$ =225.4 | $\mu=\bar{y}_{..}$=187.7 |
| Col Effect | $\alpha_1=\bar{y}_{.1}-\bar{y}_{..}$ =-13.3 | $\alpha_2=\bar{y}_{.2}-\bar{y}_{..}$ =-24.5 | $\alpha_3=\bar{y}_{.3}-\bar{y}_{..}$ =37.7 | |

# Example 20.1: Interpretation

- Average processor requires 187.7 bytes of storage.
- The effects of the processors R, V, and Z are -13.3, -24.5, and 37.7, respectively. That is,
  - R requires 13.3 bytes less than an average processor
  - V requires 24.5 bytes less than an average processor, and
  - Z requires 37.7 bytes more than an average processor.

# Estimating Experimental Errors

- Estimated response for *j*th alternative:

$$\hat{y}_j = \mu + \alpha_j$$

- Error:

$$e_{ij} = y_j - \hat{y}_j$$

- Sum of squared errors (SSE):

$$\mathrm{SSE} = \sum_{i=1}^{r} \sum_{j=1}^{a} e_{ij}^2$$

# Example 20.2

$$\begin{bmatrix} 144 & 101 & 130 \\ 120 & 144 & 180 \\ 176 & 211 & 141 \\ 288 & 288 & 374 \\ 144 & 72 & 302 \end{bmatrix} = \begin{bmatrix} 187.7 & 187.7 & 187.7 \\ 187.7 & 187.7 & 187.7 \\ 187.7 & 187.7 & 187.7 \\ 187.7 & 187.7 & 187.7 \\ 187.7 & 187.7 & 187.7 \end{bmatrix}$$

$$+ \begin{bmatrix} -13.3 & -24.5 & 37.7 \\ -13.3 & -24.5 & 37.7 \\ -13.3 & -24.5 & 37.7 \\ -13.3 & -24.5 & 37.7 \\ -13.3 & -24.5 & 37.7 \end{bmatrix}$$

$$+ \begin{bmatrix} -30.4 & -62.2 & -95.4 \\ -54.4 & -19.2 & -45.4 \\ 1.6 & 47.8 & -84.4 \\ 113.6 & 124.8 & 148.6 \\ -30.4 & -91.2 & 76.6 \end{bmatrix}$$

$$\text{SSE} = (-30.4)^2 + (-54.4)^2 + \cdots + (76.6)^2 = 94365.20$$

# Allocation of Variation

$$y_{ij} = \mu + \alpha_j + e_{ij}$$

$$y_{ij}^2 = \mu^2 + \alpha_j^2 + e_{ij}^2 + 2\mu\alpha_j + 2\mu e_{ij} + 2\alpha_j e_{ij}$$

$$\sum_{i,j} y_{ij}^2 \quad = \quad \sum_{i,j} \mu^2 + \sum_{i,j} \alpha_j^2 + \sum_{i,j} e_{ij}^2$$
$$+\text{Cross product terms}$$

$$\text{SSY} = \text{SS0} + \text{SSA} + \text{SSE}$$

$$\text{SS0} = \sum_{i=1}^{r} \sum_{j=1}^{a} \mu^2 = ar\mu^2$$

# Allocation of Variation (Cont)

$$\begin{aligned} \text{SSA} &= \sum_{i=1}^{r}\sum_{j=1}^{a}\alpha_j^2 \\ &= r\sum_{j=1}^{a}\alpha_j^2 \end{aligned}$$

- Total variation of y (SST):

$$\begin{aligned} \text{SST} &= \sum_{i,j}(y_{i,j} - \bar{y}_{..})^2 \\ &= \sum_{i,j} y_{ij}^2 - ar\bar{y}_{..}^2 \\ &= \text{SSY} - \text{SS0} = \text{SSA} + \text{SSE} \end{aligned}$$

# Example 20.3

$$
\begin{aligned}
\mathrm{SSY} &= 144^2 + 120^2 + \cdots + 302^2 = 633639 \\
\mathrm{SS0} &= ar\mu^2 \\
&= 3 \times 5 \times (187.7)^2 = 528281.7 \\
\mathrm{SSA} &= r \sum_j \alpha_j^2 \\
&= 5[(-13.3)^2 + (-24.5)^2 + (37.6)^2] \\
&= 10992.1 \\
\mathrm{SST} &= \mathrm{SSY} - \mathrm{SS0} \\
&= 633639.0 - 528281.7 = 105357.3 \\
\mathrm{SSE} &= \mathrm{SST} - \mathrm{SSA} \\
&= 105357.3 - 10992.1 = 94365.2
\end{aligned}
$$

# Example 20.3 (Cont)

$$\text{Percent variation explained by processors} = 100 \times \frac{10992.13}{105357.3} = 10.4\%$$

- 89.6% of variation in code size is due to experimental errors (programmer differences).

  Is 10.4% statistically significant?

# Analysis of Variance (ANOVA)

- Importance $\neq$ Significance
- Important $\Rightarrow$ Explains a high percent of variation
- Significance
  $\Rightarrow$ High contribution to the variation compared to that by errors.
- Degree of freedom
  = Number of independent values required to compute

$$
\begin{array}{ccccccccc}
\text{SSY} & = & \text{SS0} & + & \text{SSA} & + & \text{SSE} \\
\text{ar} & = & 1 & + & (\text{a-1}) & + & \text{a(r-1)}
\end{array}
$$

Note that the degrees of freedom also add up.

# F-Test

- Purpose: To check if SSA is *significantly* greater than SSE.

  Errors are normally distributed $\Rightarrow$ SSE and SSA have chi-square distributions.

  The ratio $(SSA/\nu_A)/(SSE/\nu_e)$ has an F distribution.

  where $\nu_A$=a-1 = degrees of freedom for SSA

  $\nu_e$=a(r-1) = degrees of freedom for SSE

  Computed ratio > $F_{[1-\alpha; \nu_A, \nu_e]}$

  $\Rightarrow$ SSA is significantly higher than SSE.

  $SSA/\nu_A$ is called mean square of A or (MSA).

  Similary, $MSE=SSE/\nu_e$

# ANOVA Table for One Factor Experiments

| Component | Sum of Squares | %Variation | DF | Mean Square | F-Comp. | F-Table |
|---|---|---|---|---|---|---|
| y | SSY=$\sum y_{ij}^2$ | | ar | | | |
| $\bar{y}_{..}$ | SS0=$ar\mu^2$ | | 1 | | | |
| y-$\bar{y}_{..}$ | SST=SSY-SS0 | 100 | ar-1 | | | |
| A | SSA $= r\Sigma\ \alpha_i^2$ | $100\left(\frac{\text{SSA}}{\text{SST}}\right)$ | a-1 | MSA $= \frac{\text{SSA}}{a-1}$ | $\frac{\text{MSA}}{\text{MSE}}$ | $F_{[1-\alpha; a-1, a(r-1)]}$ |
| e | SSE=SST- SSA | $100\left(\frac{\text{SSE}}{\text{SST}}\right)$ | $a(r-1)$ | MSE $= \frac{\text{SSE}}{a(r-1)}$ | | |

# Example 20.4: Code Size Comparison

| Component | Sum of Squares | %Variation | DF | Mean Square | F-Comp. | F-Table |
|---|---|---|---|---|---|---|
| y | 633639.00 | | | | | |
| $\bar{y}_{..}$ | 528281.69 | | | | | |
| y-$\bar{y}_{..}$ | 105357.31 | 100.0% | 14 | | | |
| A | 10992.13 | 10.4% | 2 | 5496.1 | 0.7 | 2.8 |
| Errors | 94365.20 | 89.6% | 12 | 7863.8 | | |

$$s_e = \sqrt{\text{MSE}} = \sqrt{7863.77} = 88.68$$

- Computed F-value < F from Table
  - The variation in the code sizes is mostly due to experimental errors and not because of any significant difference among the processors.

# Visual Diagnostic Tests

**Assumptions**:

1. Factors effects are additive.
2. Errors are additive.
3. Errors are independent of factor levels.
4. Errors are normally distributed.
5. Errors have the same variance for all factor levels.

**Tests**:

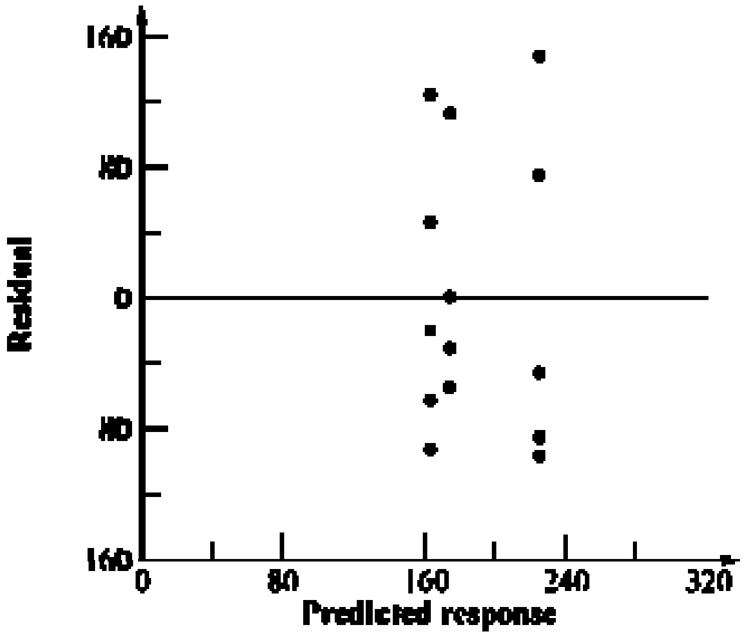
- Residuals versus predicted response:

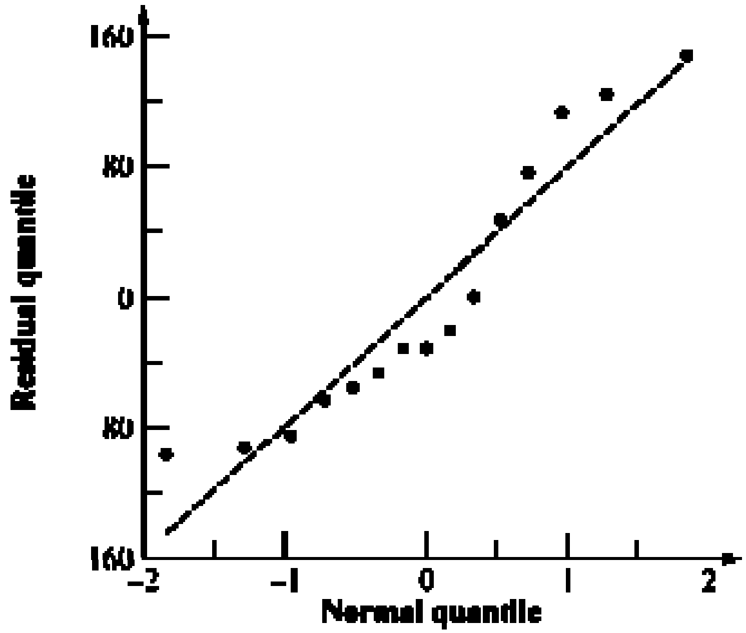
  No trend $\Rightarrow$ Independence

  Scale of errors << Scale of response

  $\Rightarrow$ Ignore visible trends.
- Normal quantilte-quantile plot linear $\Rightarrow$ Normality

# Example 20.5




- Horizontal and vertical scales similar

  ⇒ Residuals are not small ⇒ Variation due to factors is small compared to the unexplained variation
- No visible trend in the spread
- Q-Q plot is S-shaped ⇒ shorter tails than normal.

# Confidence Intervals For Effects

- Estimates are random variables

| Parameter | Estimate | Variance |
|---|---|---|
| $\mu$ | $\bar{y}_{..}$ | $s_e^2/ar$ |
| $\alpha_j$ | $\bar{y}_{.j}$-$\bar{y}_{..}$ | $s_e^2(a-1)/ar$ |
| $\mu$+$\alpha_j$ | $\bar{y}_{.j}$ | $s_e^2/r$ |
| $\sum_{j=1}^a h_j\alpha_j, \sum_{j=1}^a h_j$=0 | $\sum_{j=1}^a h_j\ \bar{y}_{.j}$ | $\sum_{j=1}^a s_e^2\ h_j^2/r$ |
| $s_e^2$ | $\frac{\sum e_{ij}^2}{a(r-1)}$ | |

Degrees of freedom for errors = a(r-1)

- For the confidence intervals, use t values at a(r-1) degrees of freedom.
- Mean responses: $\hat{y}_j = \mu + \alpha_j$
- Contrasts $\sum h_j\ \alpha_j$: Use for $\alpha_1$-$\alpha_2$

# Example 20.6: Code Size Comparison

Error variance $s_e^2 = \frac{94365.2}{12} = 7863.8$

$$\begin{aligned}\text{Std Dev of errors} &= \sqrt{(\text{Var. of errors})} \\ &= 88.7\end{aligned}$$

Std Dev of $\mu = s_e/\sqrt{ar} = 88.7/\sqrt{15} = 22.9$

$$\begin{aligned}\text{Std Dev of } \alpha_j &= s_e\sqrt{\{(a-1)/(ar)\}} \\ &= 88.7\sqrt{(2/15)} = 32.4\end{aligned}$$

# Example 20.6 (Cont)

- For 90% confidence, $t_{[0.95;\,12]} = 1.782$.
- 90% confidence intervals:

$$\begin{aligned}
\mu &= 197.7 \mp (1.782)(22.9) = (146.9, 228.5) \\
\alpha_1 &= -13.3 \mp (1.782)(32.4) = (-71.0, 44.4) \\
\alpha_2 &= -24.5 \mp (1.782)(32.4) = (-82.2, 33.2) \\
\alpha_3 &= 37.6 \mp (1.782)(32.4) = (-20.0, 95.4)
\end{aligned}$$

- The code size on an average processor is significantly different from zero.
- Processor effects are not significant.

# Example 20.6 (Cont)

- Using $h_1=1$, $h_2=-1$, $h_3=0$, ($\sum h_j=0$):

$$\text{Mean } \alpha_1 - \alpha_2 = \bar{y}_{.1} - \bar{y}_{.2} = 174.4 - 163.2 = 11.2$$

$$\begin{aligned}\text{Std dev of } \alpha_1 - \alpha_2 &= s_e\sqrt{\left(\sum h_j^2/r\right)} \\ &= 88.7\sqrt{(2/5)} = 56.1\end{aligned}$$

$$\begin{aligned}90\% \text{ CI for } \alpha_1 - \alpha_2 &= 11.2 \mp (1.782)(56.1) \\ &= (-88.7,\ 111.1)\end{aligned}$$

- CI includes zero $\Rightarrow$ one isn't superior to other.

# Example 20.6 (Cont)

- Similarly,

$$90\% \text{ CI for } \alpha_1 - \alpha_3$$
$$= (174.4 - 225.4) \mp (1.782)(56.1)$$
$$= (-150.9,\ 48.9)$$

$$90\% \text{ CI for } \alpha_2 - \alpha_3$$
$$= (163.2 - 225.4) \mp (1.782)(56.1)$$
$$= (-162.1,\ 37.7)$$

- Any one processor is not superior to another.

# Unequal Sample Sizes

$$y_{ij} = \mu + \alpha_j + e_{ij}$$

- By definition:

$$\sum_{j=1}^{a} r_j \alpha_j = 0$$

- Here, $r_j$ is the number of observations at *j*th level.
  N =total number of observations:

$$N = \sum_{j=1}^{a} r_j$$

# Parameter Estimation

| Parameter | Estimate | Variance |
|---|---|---|
| $\mu$ | $\bar{y}_{..}$ | $s_e^2/N$ |
| $\alpha_j$ | $\bar{y}_{.j}$-$\bar{y}_{..}$ | $s_e^2(N-r_j)/(Nr_j)$ |
| $\mu$+$\alpha_j$ | $\bar{y}_{.j}$ | $s_e^2/r_j$ |
| $\Sigma\ h_j\alpha_j,\ \Sigma\ h_j$=0 | $h_j\ \bar{y}_{.j}$ | $s_e^2\sum_{j=1}^{a}\ (h_j^2/r_j)$ |
| $s_e^2$ | $\sum\ e_{ij}^2/\{N-a\}$ | |

Degrees of freedom for errors = N-a

# Analysis of Variance

| Component | Sum of Squares | %Variation | DF | Mean Square | F-Comp. | F-Table |
|---|---|---|---|---|---|---|
| y | SSY=$\sum y_{ij}^2$ | | N | | | |
| $\bar{y}_{..}$ | SS0=$N\mu^2$ | | 1 | | | |
| y-$\bar{y}_{..}$ | SST=SSY-SS0 | 100 | N-1 | | | |
| A | SSA $= \sum_{j=1}^{a} r_j \alpha_j^2$ | $100 \left(\frac{\text{SSA}}{\text{SST}}\right)$ | a-1 | MSA=$\frac{\text{SSA}}{a-1}$ | $\frac{\text{MSA}}{\text{MSE}}$ | $F_{[1-\alpha;a-1,N-a]}$ |
| e | SSE=SST- SSA | $100 \left(\frac{\text{SSE}}{\text{SST}}\right)$ | N-a | MSE=$\frac{\text{SSE}}{N-a}$ | | |

# Example 20.7: Code Size Comparison

| | R | V | Z | |
|---|---|---|---|---|
| | 144 | 101 | 130 | |
| | 120 | 144 | 180 | |
| | 176 | 211 | 141 | |
| | 288 | 288 | | |
| | 144 | | | |
| Column Sum | 872 | 744 | 451 | 2067 |
| Column Mean | 174.40 | 186.00 | 150.33 | 172.25 |
| Column effect | 2.15 | 13.75 | -21.92 | |

- All means are obtained by dividing by the number of observations added.
- The column effects are 2.15, 13.75, and -21.92.

# Example 20.6: Analysis of Variance

$$
\begin{bmatrix} 144 & 101 & 130 \\ 120 & 144 & 180 \\ 176 & 211 & 141 \\ 288 & 288 & \\ 144 & & \end{bmatrix}
=
\begin{bmatrix} 172.25 & 172.25 & 172.25 \\ 172.25 & 172.25 & 172.25 \\ 172.25 & 172.25 & 172.25 \\ 172.25 & 172.25 & \\ 172.25 & & \end{bmatrix}
+
\begin{bmatrix} 2.15 & 13.75 & -21.92 \\ 2.15 & 13.75 & -21.92 \\ 2.15 & 13.75 & -21.92 \\ 2.15 & 13.75 & \\ 2.15 & & \end{bmatrix}
$$

$$
+
\begin{bmatrix} -30.40 & -85.00 & -20.33 \\ -54.40 & -42.00 & 29.67 \\ 1.60 & 25.00 & -9.33 \\ 113.60 & 102.00 & \\ -30.40 & & \end{bmatrix}
$$

# Example 20.6 ANOVA (Cont)

- Sums of Squares:

$$
\begin{aligned}
\text{SSY} &= \sum y_{ij}^2 = 397375 \\
\text{SS0} &= N\mu^2 = 356040.75 \\
\text{SSA} &= 5\alpha_1^2 + 4\alpha_2^2 \\
&\quad +3\alpha_3^2 = 2220.38 \\
\text{SSE} &= (-30.40)^2 + (-54.40)^2 + \cdots \\
&\quad +(-9.33)^2 = 39113.87 \\
\text{SST} &= \text{SSY} - \text{SS0} = 41334.25
\end{aligned}
$$

- Degrees of Freedom:

$$
\begin{array}{ccccccccc}
\text{SSY} & = & \text{SS0} & + & \text{SSA} & + & \text{SSE} \\
\text{N} & = & 1 & + & \text{(a-1)} & + & \text{N-a} \\
12 & = & 1 & + & 2 & + & 9
\end{array}
$$

# Example 20.6 ANOVA Table

| Component | Sum of Squares | %Variation | DF | Mean Square | F-Comp. | F-Table |
|---|---|---|---|---|---|---|
| y | 397375.00 | | | | | |
| $y_{..}$ | 356040.75 | | | | | |
| y-$y_{..}$ | 41334.25 | 100.00% | 11 | | | |
| A | 2220.38 | 5.37% | 2 | 1110.19 | 0.26 | 3.01 |
| Errors | 39113.87 | 94.63% | 9 | 4345.99 | | |

$$s_e = \sqrt{\text{MSE}} = \sqrt{4345.99} = 65.92$$

- **Conclusion**: Variation due processors is insignificant as compared to that due to modeling errors.

# Example 20.6 Standard Dev. of Effects

- Consider the effect of processor Z: Since,

$$
\begin{aligned}
\alpha_3 &= y_{.3} - y_{..} \\
&= \frac{1}{3}(y_{13} + y_{23} + y_{33}) - \frac{1}{12}(y_{11} \\
&\quad + y_{21} + \cdots + y_{32} + y_{42} + y_{13} + y_{23} + y_{33}) \\
&= \frac{1}{4}(y_{13} + y_{23} + y_{33}) - \frac{1}{12}(y_{11} + y_{21} + \cdots + y_{32} + y_{42})
\end{aligned}
$$

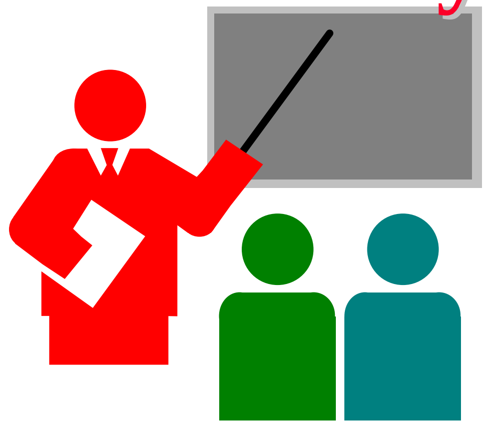
- Error in $\alpha_3 = \sum$ Errors in terms on the right hand side:

$$
e_{\alpha_3} = \frac{1}{4}(e_{13} + e_{23} + e_{33}) - \frac{1}{12}(e_{11} + e_{21} + \cdots + e_{32} + e_{42})
$$

- $e_{ij}$'s are normally distributed $\Rightarrow \alpha_3$ is normal with

$$
s_{\alpha_3}^2 = \frac{1}{4^2} \times 3s_e^2 + \frac{1}{12^2} \times 9s_e^2 = 1086.36
$$

# Summary



- Model for One factor experiments:

$$y_{ij} = \mu + \alpha_j + e_{ij} \qquad \sum_{j=1}^{a} \alpha_j = 0$$

- Computation of effects
- Allocation of variation, degrees of freedom
- ANOVA table
- Standard deviation of errors
- Confidence intervals for effects and contracts
- Model assumptions and visual tests

# Exercise 20.1

For a single factor design, suppose we want to write an expression for $\alpha_j$ in terms of $y_{ij}$'s:

$$\alpha_j = a_{11j}y_{11} + a_{12j}y_{12} + \cdots + a_{raj}y_{ra}$$

What are the values of $a_{..j}$'s? From the above expression, the error in $\alpha_j$ is seen to be:

$$e_{\alpha_j} = a_{11j}e_{11} + a_{12j}e_{12} + \cdots + a_{raj}e_{ra}$$

Assuming errors $e_{ij}$ are normally distributed with zero mean and variance $\sigma_e^2$, write an expression for variance of $e_{\alpha_j}$. Verify that your answer matches that in Table 20.5.

# Homework

Analyze the following one factor experiment:

| R | V | Z |
|---|---|---|
| 145 | 102 | 131 |
| 120 | 144 | 180 |
| 177 | 212 | 142 |
| 288 | | |
| 144 | | |

1. Compute the effects
2. Prepare ANOVA table
3. Compute confidence intervals for effects and interpret
4. Compute Confidence interval for $\alpha_1$-$\alpha_3$
5. Show graphs for visual tests and interpret