Other Public-Key Cryptosystems

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Audio/Video recordings of this lecture are available at:

http://www.cse.wustl.edu/~jain/cse571-17/

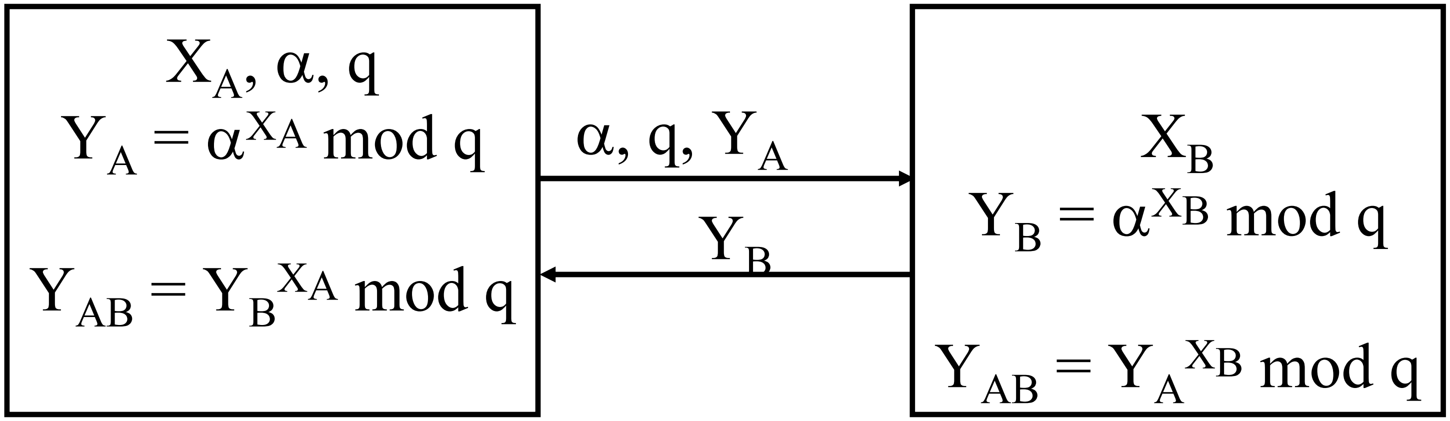
Overview

1. How to exchange keys in public? (Diffie-Hellman Key Exchange)
2. ElGamal Cryptosystem
3. Elliptic Curve Arithmetic
4. Elliptic Curve Cryptography
5. Pseudorandom Number Generation using Asymmetric Cipher

These slides are based partly on Lawrie Brown's slides supplied with William Stallings's book "Cryptography and Network Security: Principles and Practice," 7th Ed, 2013.

Diffie-Hellman Key Agreement

- Allows two party to agree on a secret key using a public channel
- A selects q=large prime, and α=a primitive root of q
- A selects a random # X_A, B selects another random # X_B



X_A, α, q

$Y_A = \alpha^{X_A} \bmod q$

$Y_{AB} = Y_B^{X_A} \bmod q$

α, q, Y_A

Y_B

X_B

$Y_B = \alpha^{X_B} \bmod q$

$Y_{AB} = Y_A^{X_B} \bmod q$

$$Y_{AB} = \alpha^{X_A X_B} \bmod q$$

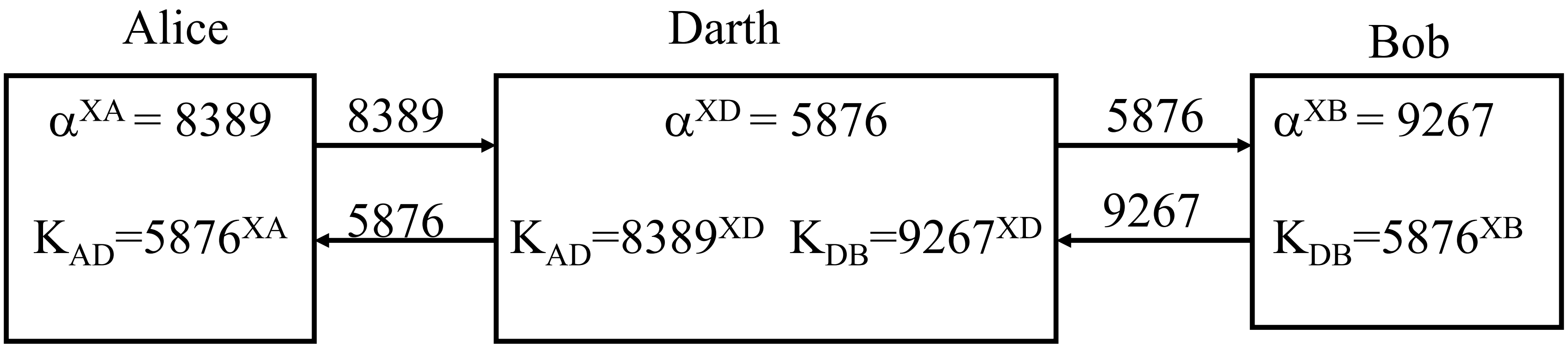
- Eavesdropper can see Y_A, α, q but cannot compute X_A
- Computing X_A requires discrete logarithm - a difficult problem

Diffie-Hellman (Cont)

- Example: α=5, q=19
  - A selects 6 and sends 5^6 mod 19 = 7
  - B selects 7 and sends 5^7 mod 19 = 16
  - A computes K = 16^6 mod 19 = 7
  - B computes K = 7^7 mod 19 = 7
- Preferably (q-1)/2 should also be a prime.
- Such primes are called safe prime.

Man-in-Middle Attack on Diffie-Hellman

- Diffie-Hellman does not provide authentication



Alice

$\alpha^{XA} = 8389$

$K_{AD}=5876^{XA}$

8389

5876

Darth

$\alpha^{XD} = 5876$

$K_{AD}=8389^{XD}$ $K_{DB}=9267^{XD}$

5876

9267

Bob

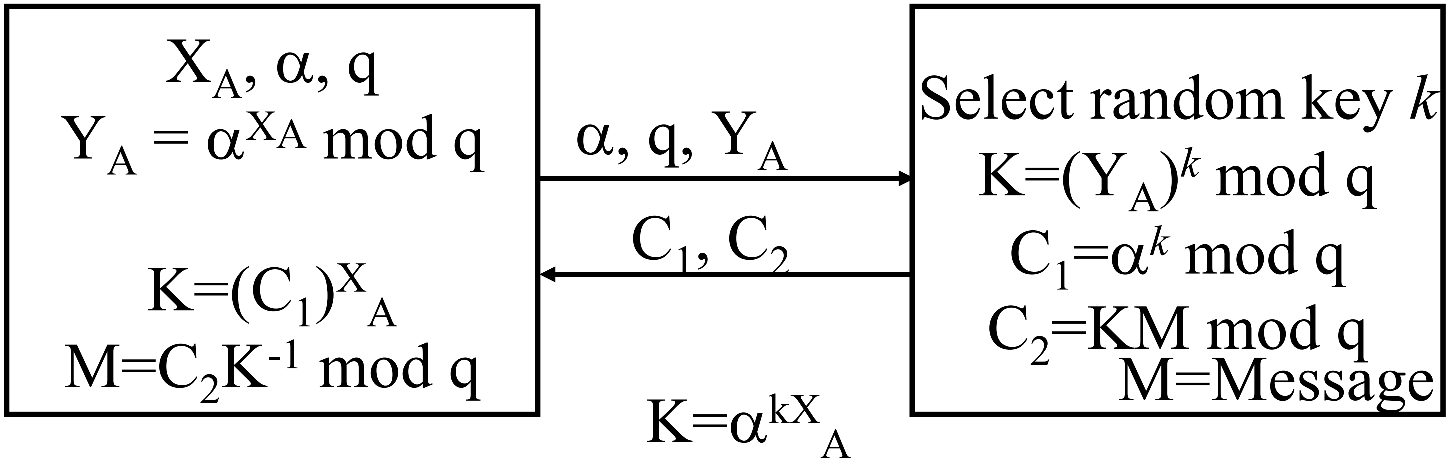
$\alpha^{XB} = 9267$

$K_{DB}=5876^{XB}$

- X can then intercept, decrypt, re-encrypt, forward all messages between Alice & Bob
- You can use RSA authentication and other alternatives

ElGamal Cryptography

- Public-key cryptosystem related to D-H
- Uses exponentiation in a finite (Galois)
- Security based difficulty of computing discrete logarithms
- X_A is the private key, $\{\alpha, q, Y_{A\}}$ is the public key



X_A, α, q

$Y_A = \alpha^{X_A} \bmod q$

$K=(C_1)^{X_A}$

$M=C_2K^{-1} \bmod q$

α, q, Y_A

C_1, C_2

Select random key k

$K=(Y_A)^k \bmod q$

$C_1=\alpha^k \bmod q$

$C_2=KM \bmod q$

M=Message

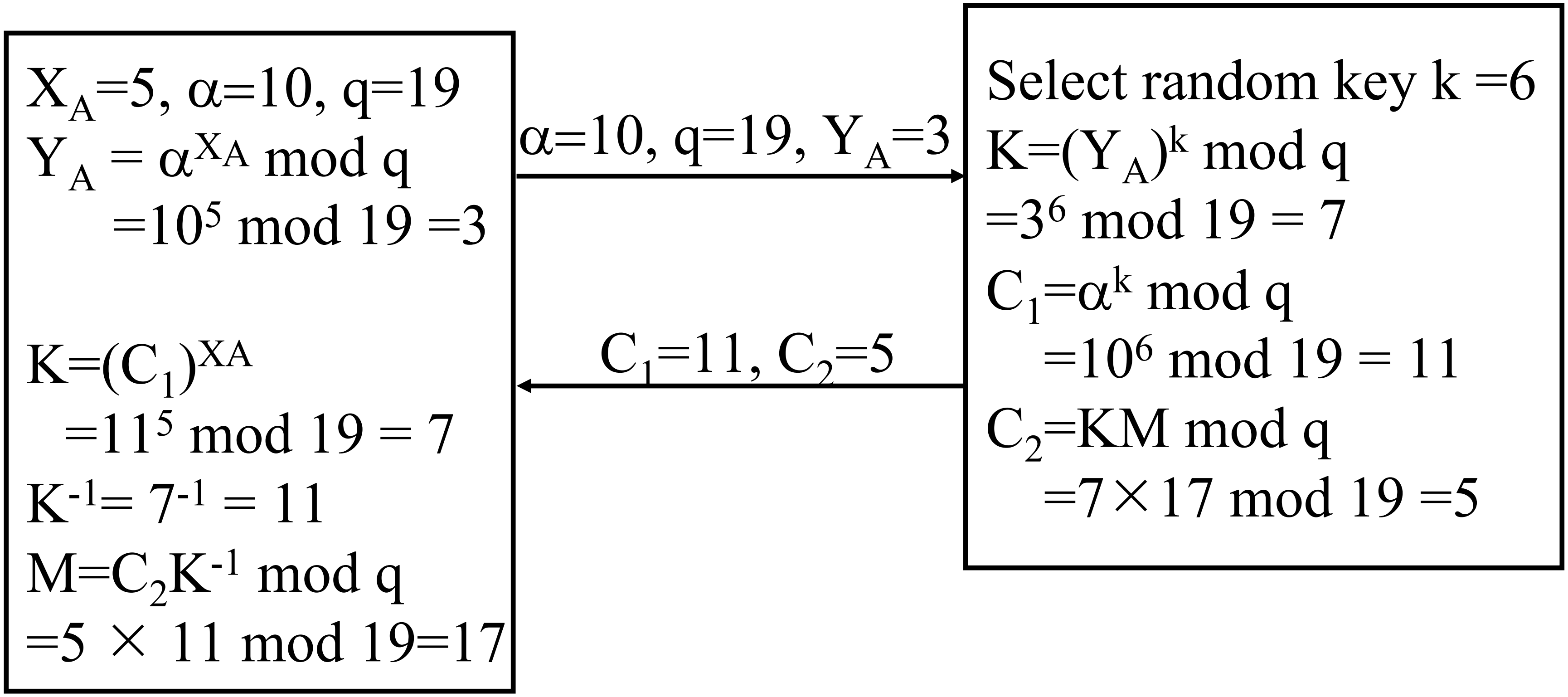
$K=\alpha^{kX_A}$

- k must be unique each time. Otherwise insecure.

Ref: http://en.wikipedia.org/wiki/ElGamal_encryption

ElGamal Cryptography Example

- Use field GF(19) q=19 and α=10
- Alice chooses $x_A=5$,
- Bob wants to sent message M=17, selects a random key k=6



$X_A=5, \alpha=10, q=19$

$Y_A = \alpha^{XA} \bmod q$

$=10^5 \bmod 19 = 3$

$K=(C_1)^{XA}$

$=11^5 \bmod 19 = 7$

$K^{-1}= 7^{-1} = 11$

$M=C_2K^{-1} \bmod q$

$=5 \times 11 \bmod 19=17$

$\alpha=10, q=19, Y_A=3$

$C_1=11, C_2=5$

Select random key k =6

$K=(Y_A)^k \bmod q$

$=3^6 \bmod 19 = 7$

$C_1=\alpha^k \bmod q$

$=10^6 \bmod 19 = 11$

$C_2=KM \bmod q$

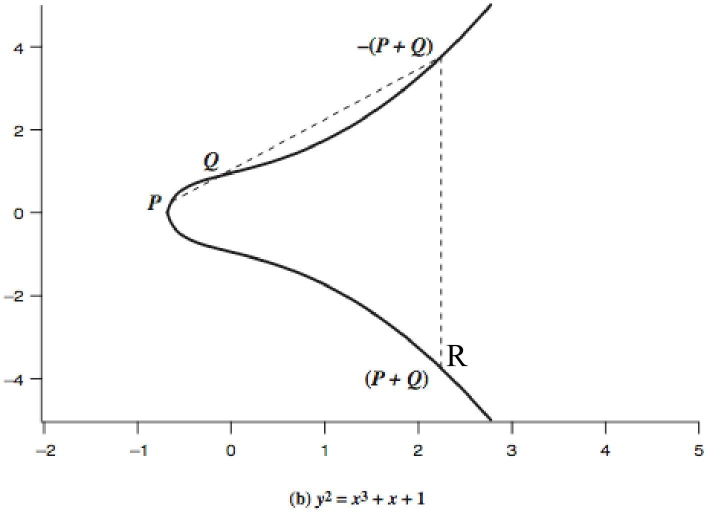
$=7\times 17 \bmod 19 =5$

Elliptic Curve Cryptography

- Majority of public-key crypto (RSA, D-H) use either integer or polynomial arithmetic with very large numbers/polynomials
- Imposes a significant load in storing and processing keys and messages
- An alternative is to use elliptic curves
- Offers same security with smaller bit sizes
- Newer, but not as well analyzed

Elliptic Curves over Real Numbers

- An elliptic curve is defined by an equation in two variables x & y,
  - $y^2 = x^3 + ax + b$
  - Where x, y, a, b are all real numbers
  - $4a^3+27b^2\neq 0$
- The set of points E(a, b) forms an abelian group with respect to "addition" operation defined as follows:
  - P+Q is reflection of the intersection R
  - O (Infinity) acts as additive identity
  - To double a point P, find intersection of tangent and curve
  - Closure: P+Q ε E
  - Associativity: P+(Q+R) = (P+Q)+R
  - Identity: P+O=P
  - Inverse: -P ε E
  - Commutative: P+Q = Q+P

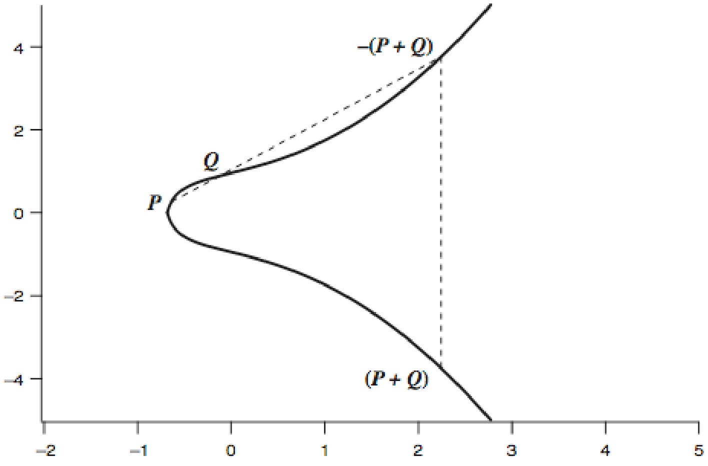


Elliptic Curve over Real Numbers (Cont)

- Slope of line PQ is:
  - $\Delta = (y_Q - y_P)/(x_Q - x_P)$
- The sum R=P+Q is:
  - $x_R = \Delta^2 - x_P - x_Q$
  - $y_R = -y_p + \Delta(x_P - x_R)$
- P+P=2P=R

$$x_R = \left(\frac{3x_P^2 + a}{2y_P}\right)^2 - 2x_P$$

$$y_r = \left(\frac{3x_P^2 + a}{2y_P}\right)(x_P - x_R) - y_P$$



(b) $y^2 = x^3 + x + 1$

Finite Elliptic Curves

- Elliptic curve cryptography uses curves whose variables & coefficients are defined over GF
  - **Prime curves**: $E_p(a,b)$ defined over Z_p
    - Use integers modulo a prime
    - Easily implemented in software
  - **Binary curves**: $E_{2m}(a,b)$ defined over $GF(2^n)$
    - Use polynomials with binary coefficients
    - Easily implemented in hardware
- Cryptography: Addition in elliptic = multiplication in Integer
  - Repeated addition = Exponentiation
  - Easy to compute Q=P+P+…+P=kP, where Q, P ε E
  - Hard to find k given Q, P (Similar to discrete log)

Finite Elliptic Curve Example

- $E_p(a,b)$: $y^2=x^3+ax+b \bmod p$
- $E_{23}(1,1)$: $y^2=x^3+x+1 \bmod 23$
- Consider only +ve x and y
- R=P+Q
  - $x_R=(\lambda^2-x_P-x_Q) \bmod p$
  - $y_R=(\lambda(x_P-x_R)-y_P) \bmod p$
  - Where

$$\lambda = \begin{cases} \left(\frac{y_Q-y_P}{x_Q-x_P}\right) \bmod p & \text{if } P \neq Q \\ \left(\frac{3x_P^2+a}{2y_P}\right) \bmod p & \text{if } P = Q \end{cases}$$

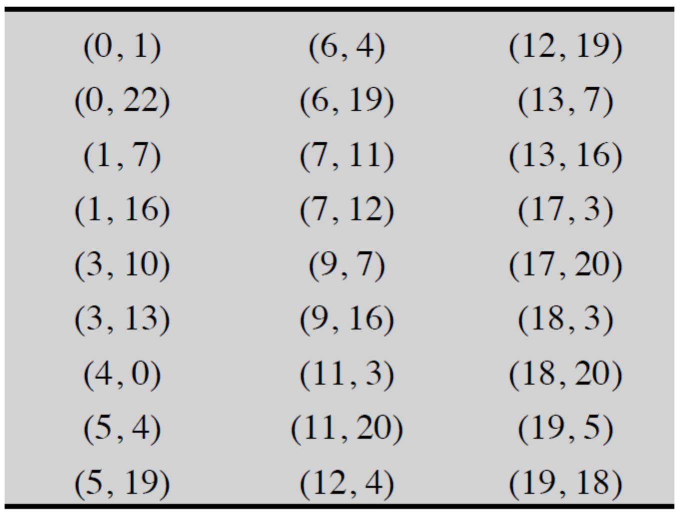
- Example: (3,10)+(3,10)

$$\lambda = \left(\frac{3(3^2)+1}{2\times 10}\right) \bmod 23 = \frac{1}{4} \bmod 23 = 6$$

$x_R=(6^2-3-3) \bmod 23 = 7$

$y_R=(6(3-7)-10) \bmod 23 = 12$

Table 10.1 Points on the Elliptic Curve $E_{23}(1, 1)$



(0, 1)	(6, 4)	(12, 19)
(0, 22)	(6, 19)	(13, 7)
(1, 7)	(7, 11)	(13, 16)
(1, 16)	(7, 12)	(17, 3)
(3, 10)	(9, 7)	(17, 20)
(3, 13)	(9, 16)	(18, 3)
(4, 0)	(11, 3)	(18, 20)
(5, 4)	(11, 20)	(19, 5)
(5, 19)	(12, 4)	(19, 18)

ECC Example

- $E_{211}(0, -4)$, $y^2=x^3+ax+b=x^3-4$
- G=(2,2), Calculate 121G
- 121=1111001 ⇒ 64G+32G+16G+8G+G

$$
\begin{aligned}
\lambda &= \frac{y_Q - y_P}{x_q - x_P} \text{ or } \frac{3x_P^2 + a}{2y_P} \\
R &= P + Q = \left[\lambda^2 - x_P - x_Q, \lambda(x_P - x_R) - y_P\right]
\end{aligned}
$$

Point	①	②	③	λ	x_R	y_R
G					2	2
2G=G+G	12	4	53	3	5	200
4G=2G+2G	75	189	163	198	159	114
8G=4G+4G	94	17	149	80	174	163
16G=8G+8G	98	115	200	188	181	209
32G=16G+16G	168	207	158	169	136	11
64G=32G+32G	206	22	48	182	147	97
9G=G+8G	161	172	119	169	111	145
25G=9G+16G	64	70	208	19	69	20
57G=25G+32G	202	67	63	66	142	15
121G=57G+64G	82	5	169	143	115	48

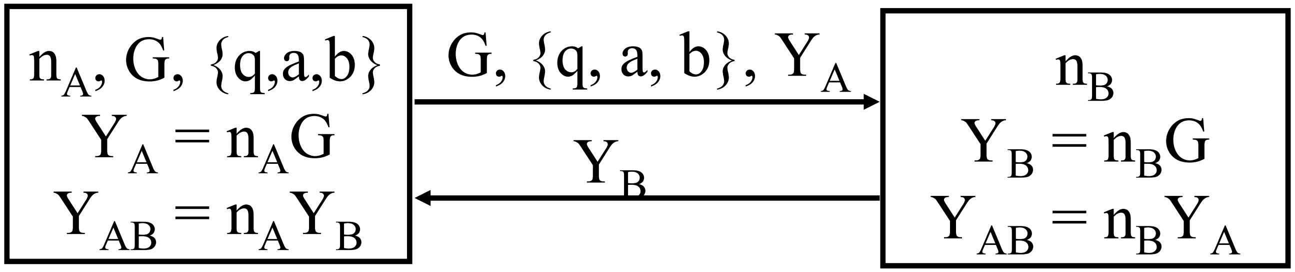
① Numerator of λ

② Denominator of λ

③ (Denominator of λ)$^{-1}$

ECC Diffie-Hellman

- Select a suitable curve $E_q(a,b)$
- Select base point $G=(x_1,y_1)$ with large order n s.t. $nG=0$
- A & B select private keys $n_A<n,\ \ n_B<n$
- Compute public keys: $Y_A=n_AG,\ \ Y_B=n_BG$
- Compute shared key: $K=n_AY_B,\ \ K=n_BY_A$
  - Same since $K=n_An_BG$
- Attacker would need to find K, hard



n_A, G, $\{q,a,b\}$

$Y_A = n_AG$

$Y_{AB} = n_AY_B$

G, $\{q, a, b\}$, Y_A

Y_B

n_B

$Y_B = n_BG$

$Y_{AB} = n_BY_A$

$$Y_{AB} = n_A\, n_BG$$

ECC Encryption/Decryption

- Several alternatives, will consider simplest
- Select suitable curve & point G
- Encode any message M as a point on the elliptic curve P_m
- Each user chooses private key $n_A<n$
- Computes public key $P_A=n_AG$, $P_B=n_BG$
- Encrypt P_m : $C_m=\{kG, P_m+kP_B\}$, k random
- Decrypt C_m compute:

$$P_m+kP_B-n_B(kG) = P_m+k(n_BG)-n_B(kG) = P_m$$

ECC Encryption/Decryption Example

- $E_{257}(0, -4)$, $P_m=(112,26)$, $n_B=101$ $G=(2, 2)$
- $P_B=n_BG = 101(2, 2) =(197, 167)$
- $k=41$, $C_1=kG=41(2,2)=(136, 128)$
- $C_2=P_m+kP_B= (112, 26) + 41(197, 167)$ $=(112, 26)+(68, 84) = (246, 174)$
- $C_m=\{C_1, C_2\} = \{(136,128),(246, 174)\}$
- $P_m=C_2-n_BC_1 = (246, 174)-101(136, 128)$ $=(246, 174)-(68, 84) = (112, 26)$

ECC Security

- Relies on elliptic curve logarithm problem
- Can use much smaller key sizes than with RSA etc
- For equivalent key lengths computations are roughly equivalent
- Hence for similar security ECC offers significant computational advantages

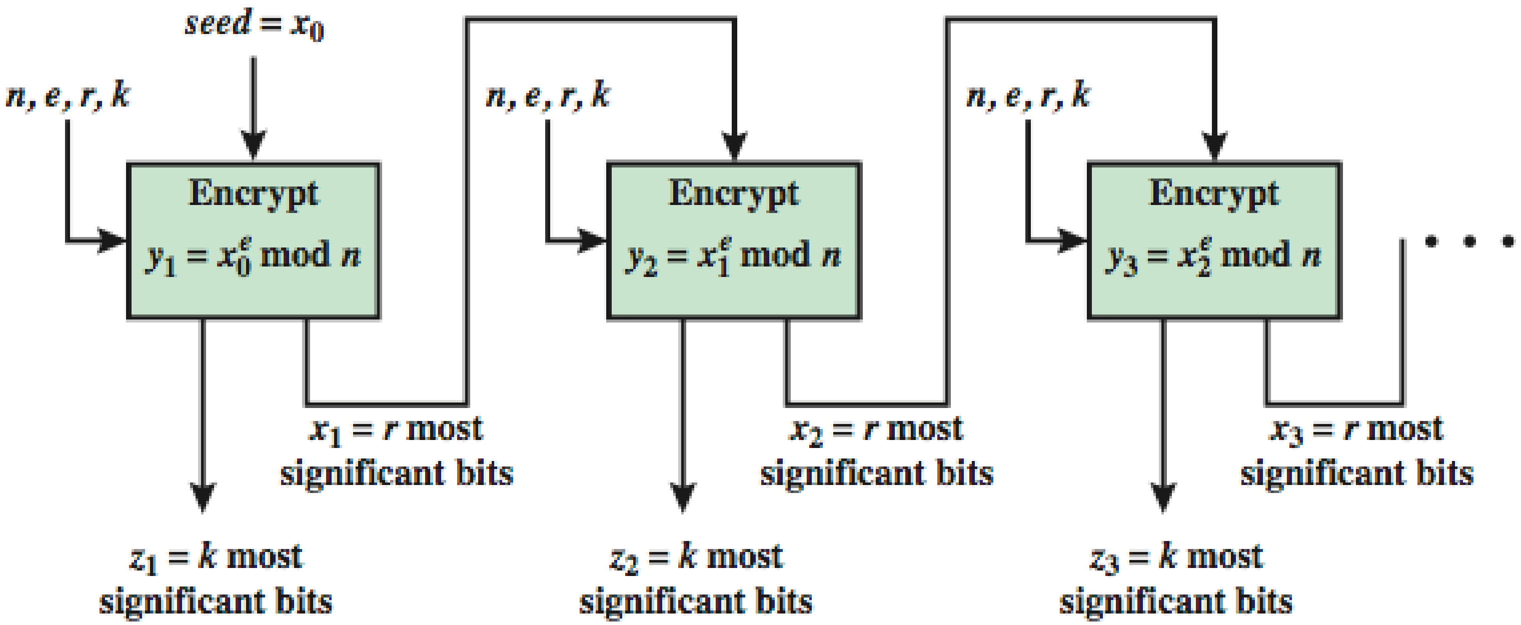
Symmetric scheme (key size in bits)	**ECC-based scheme (size of *n* in bits)**	**RSA/DSA (modulus size in bits)**
56	112	512
80	160	1024
112	224	2048
128	256	3072
192	384	7680
256	512	15360

PRNG based on Asymmetric Ciphers

- Asymmetric encryption algorithms produce apparently random output
- Hence can be used to build a pseudorandom number generator (PRNG)
- Much slower than symmetric algorithms
- Hence only use to generate a short pseudorandom bit sequence (e.g., key)

PRNG based on RSA

- Micali-Schnorr PRNG using RSA
  - in ANSI X9.82 and ISO 18031

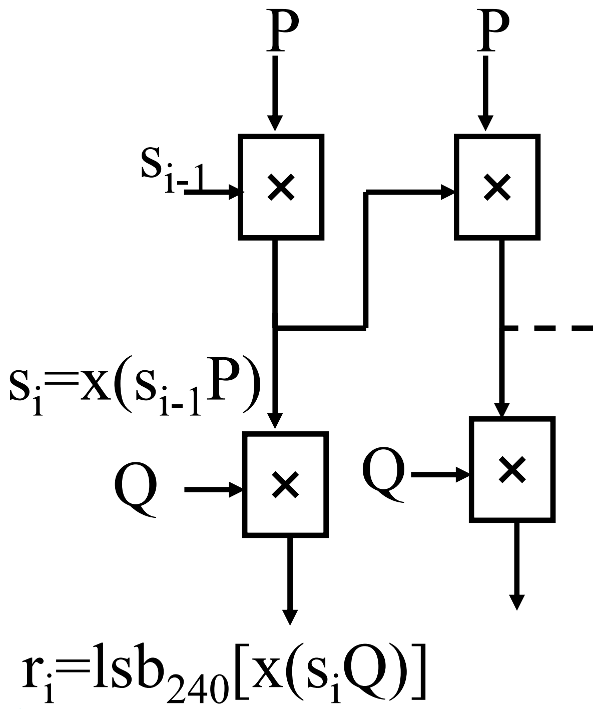

$seed = x_0$
n, e, r, k
Encrypt
$y_1 = x_0^e \bmod n$
n, e, r, k
Encrypt
$y_2 = x_1^e \bmod n$
n, e, r, k
Encrypt
$y_3 = x_2^e \bmod n$
• • •
$x_1 = r$ most significant bits
$x_2 = r$ most significant bits
$x_3 = r$ most significant bits
$z_1 = k$ most significant bits
$z_2 = k$ most significant bits
$z_3 = k$ most significant bits

PRNG based on ECC

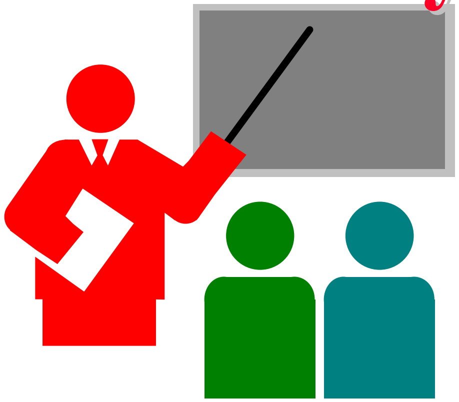
- Dual elliptic curve PRNG
  - NIST SP 800-9, ANSI X9.82 and ISO 18031
- Some controversy on security /inefficiency
- Notation: x(P) = x coordinate of P. $\text{lsb}_i(x)=i$ least sig bits of x
- Algorithm

```
for i = 1 to k do
set $s_i$ = x($s_{i-1}$ P )
set $r_i$ = $\text{lsb}_{240}$ (x($s_i$ Q))
end for
return $r_1$ , . . . , $r_k$
```

- Only use if just have ECC



Summary



1. Diffie-Hellman key exchange allows creating a secret in public based on exponentiation
2. ElGamal cryptography uses D-H
3. Elliptic Curve cryptography is based on defining addition of points on an elliptic curve in GF(p) or GF(2^n)
4. Public key cryptography (both RSA and ECC) can also be used to generate cryptographically secure pseudorandom numbers.

Homework 10

- 1. Consider an Elgamal scheme with a common prime q=71 and a primitive root α=7.
  - A. If B has public key Y_B=3 and A choose the random integer k=2, what is the ciphertext of M=30?
  - B. If A now chooses a different value of k so that the encoding of M=30 is C=(59,C_2). What is the integer C_2?
- 2. For an elliptic curve cryptography using $E_{11}(1,6)$ and G=(2,7). B's private key n_B=7.
  - A. Find B's Public key P_B
  - B. A wishes to encrypt the message P_m=(10, 9) and chooses the random value k=3. Determine the ciphertext C_m
  - C. Show the calculation by which B recovers P_m from C_m.

Lab 10: Kali Linux

- Prepare a bootable USB drive with Kali Linux
- See instructions at: http://docs.kali.org/downloading/kali-linux-live-usb-install
- You will need a 4GB or larger USB 3 flash drive
- Also, you will need to change the boot sequence in your computer to allow booting from the USB drive
- No other changes are required to your disk or computer.
- Explore Kali and submit the list of penetration tools available in Kali
- Note: Kali is a goddess that destroys evil

Ref: https://en.wikipedia.org/wiki/Kali_Linux

Acronyms

ANSI	American National Standards Institute
DEC	Dual Elliptic Curve
DSS	Digital Signature Standard
ECC	Elliptic curve cryptography
GF	Galvois Field
IEEE	Institute of Electrical and Electronic Engineers
ISO	International Standards Organization
MIME	Multipurpose Internet Multimedia Email
NIST	National Institute of Science and Technology
OFB	Output feedback mode
PRF	Pseudo-random function
PRNG	Pseudo-Random Number Generator
RSA	Rivest, Shamir, and Adleman
SP	Standard Practice
VPN	Virtual Private Network

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Related Modules

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CSE473S: Introduction to Computer Networks (Fall 2016),
http://www.cse.wustl.edu/~jain/cse473-16/index.html

Wireless and Mobile Networking (Spring 2016),
http://www.cse.wustl.edu/~jain/cse574-16/index.html

CSE571S: Network Security (Fall 2014),
http://www.cse.wustl.edu/~jain/cse571-14/index.html

Audio/Video Recordings and Podcasts of
Professor Raj Jain's Lectures,
https://www.youtube.com/channel/UCN4-5wzNP9-ruOzQMs-8NUw