

# **$2^k$ Factorial Designs**

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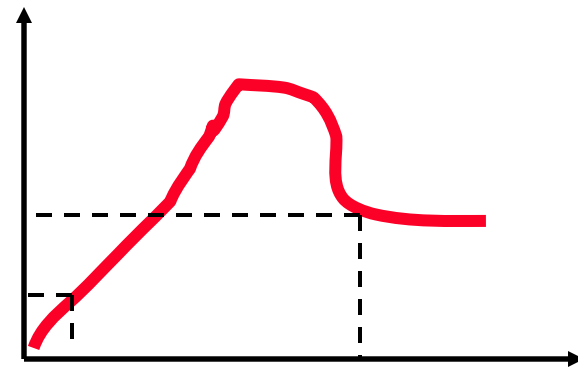
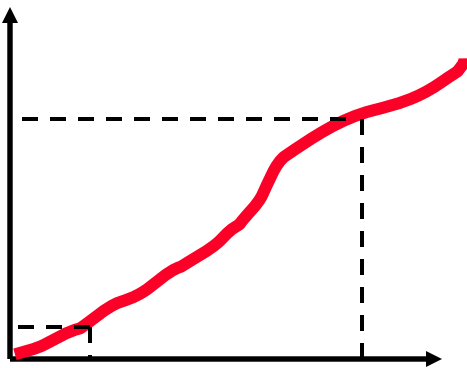
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- ❑  $2^2$  Factorial Designs
- ❑ Model
- ❑ Computation of Effects
- ❑ Sign Table Method
- ❑ Allocation of Variation
- ❑ General  $2^k$  Factorial Designs

# $2^k$ Factorial Designs

- ❑ k factors, each at two levels.
- ❑ Easy to analyze.
- ❑ Helps in sorting out impact of factors.
- ❑ Good at the beginning of a study.
- ❑ Valid only if the effect is unidirectional.  
E.g., memory size, the number of disk drives



# 2<sup>2</sup> Factorial Designs

- Two factors, each at two levels.

Performance in MIPS

Cache Size	Memory Size	
	4M Bytes	16M Bytes
1K	15	45
2K	25	75

$$x_A = \begin{cases} -1 & \text{if 4M bytes memory} \\ 1 & \text{if 16M bytes memory} \end{cases}$$
$$x_B = \begin{cases} -1 & \text{if 1K bytes cache} \\ 1 & \text{if 2K bytes cache} \end{cases}$$

## Model

$$y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B$$

Observations:

$$15 = q_0 - q_A - q_B + q_{AB}$$

$$45 = q_0 + q_A - q_B - q_{AB}$$

$$25 = q_0 - q_A + q_B - q_{AB}$$

$$75 = q_0 + q_A + q_B + q_{AB}$$

Solution:

$$y = 40 + 20x_A + 10x_B + 5x_A x_B$$

**Interpretation:** Mean performance = 40 MIPS

Effect of memory = 20 MIPS; Effect of cache = 10 MIPS

Interaction between memory and cache = 5 MIPS.

# Computation of Effects

Experiment	A	B	y
1	-1	-1	$y_1$
2	1	-1	$y_2$
3	-1	1	$y_3$
4	1	1	$y_4$

$$y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B$$

$$y_1 = q_0 - q_A - q_B + q_{AB}$$

$$y_2 = q_0 + q_A - q_B - q_{AB}$$

$$y_3 = q_0 - q_A + q_B - q_{AB}$$

$$y_4 = q_0 + q_A + q_B + q_{AB}$$

# Computation of Effects (Cont)

**Solution:**

$$q_0 = \frac{1}{4}(y_1 + y_2 + y_3 + y_4)$$

$$q_A = \frac{1}{4}(-y_1 + y_2 - y_3 + y_4)$$

$$q_B = \frac{1}{4}(-y_1 - y_2 + y_3 + y_4)$$

$$q_{AB} = \frac{1}{4}(y_1 - y_2 - y_3 + y_4)$$

Notice that effects are linear combinations of responses.

Sum of the coefficients is zero  $\Rightarrow$  **contrasts**.

## Computation of Effects (Cont)

Experiment	A	B	y
1	-1	-1	$y_1$
2	1	-1	$y_2$
3	-1	1	$y_3$
4	1	1	$y_4$

$$q_A = \frac{1}{4}(-y_1 + y_2 - y_3 + y_4)$$

$$q_B = \frac{1}{4}(-y_1 - y_2 + y_3 + y_4)$$

Notice:

$$q_A = \text{Column A} \times \text{Column y}$$

$$q_B = \text{Column B} \times \text{Column y}$$



# Sign Table Method

I	A	B	AB	y
1	-1	-1	1	15
1	1	-1	-1	45
1	-1	1	-1	25
1	1	1	1	75
160	80	40	20	Total
40	20	10	5	Total/4

# Allocation of Variation

- Importance of a factor = proportion of the *variation* explained

$$\text{Sample Variance of } y = s_y^2 = \frac{\sum_{i=1}^{2^2} (y_i - \bar{y})^2}{2^2 - 1}$$

$$\text{Total Variation of } y = \text{SST} = \sum_{i=1}^{2^2} (y_i - \bar{y})^2$$

- For a  $2^2$  design:

$$\text{SST} = 2^2 q_A^2 + 2^2 q_B^2 + 2^2 q_{AB}^2 = \text{SSA} + \text{SSB} + \text{SSAB}$$

- Variation due to A =  $\text{SSA} = 2^2 q_A^2$

- Variation due to B =  $\text{SSB} = 2^2 q_B^2$

- Variation due to interaction =  $\text{SSAB} = 2^2 q_{AB}^2$

- Fraction explained by A =  $\frac{\text{SSA}}{\text{SST}}$       Variation  $\neq$  Variance

# Derivation

□ Model:

$$y_i = q_0 + q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi}$$

Notice

1. The sum of entries in each column is zero:

$$\sum_{i=1}^4 x_{Ai} = 0; \sum_{i=1}^4 x_{Bi} = 0; \sum_{i=1}^4 x_{Ai} x_{Bi} = 0;$$

2. The sum of the squares of entries in each column is 4:

$$\sum_{i=1}^4 x_{Ai}^2 = 4$$

$$\sum_{i=1}^4 x_{Bi}^2 = 4$$

$$\sum_{i=1}^4 (x_{Ai} x_{Bi})^2 = 4$$

## Derivation (Cont)

3. The columns are orthogonal (inner product of any two columns is zero):

$$\sum_{i=1}^4 x_{Ai} x_{Bi} = 0$$

$$\sum_{i=1}^4 x_{Ai} (x_{Ai} x_{Bi}) = 0$$

$$\sum_{i=1}^4 x_{Bi} (x_{Ai} x_{Bi}) = 0$$

## Derivation (Cont)

□ Sample mean  $\bar{y}$

$$= \frac{1}{4} \sum_{i=1}^4 y_i$$

$$= \frac{1}{4} \sum_{i=1}^4 (q_0 + q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi})$$

$$= \frac{1}{4} \sum_{i=1}^4 q_0 + \frac{1}{4} q_A \sum_{i=1}^4 x_{Ai}$$

$$+ q_B \frac{1}{4} \sum_{i=1}^4 x_{Bi} + q_{AB} \frac{1}{4} \sum_{i=1}^4 x_{Ai} x_{Bi}$$

$$= q_0$$

## Derivation (Cont)

### □ Variation of y

$$\begin{aligned} &= \sum_{i=1}^4 (y_i - \bar{y})^2 \\ &= \sum_{i=1}^4 (q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi})^2 \\ &= \sum_{i=1}^4 (q_A x_{Ai})^2 + \sum_{i=1}^4 (q_B x_{Bi})^2 \\ &+ \sum_{i=1}^4 (q_{AB} x_{Ai} x_{Bi})^2 + \text{Product terms} \\ &= q_A^2 \sum_{i=1}^4 (x_{Ai})^2 + q_B^2 \sum_{i=1}^4 (x_{Bi})^2 \\ &+ q_{AB}^2 \sum_{i=1}^4 (x_{Ai} x_{Bi})^2 + 0 \\ &= 4q_A^2 + 4q_B^2 + 4q_{AB}^2 \end{aligned}$$

## Example 17.2

- Memory-cache study:

$$\bar{y} = \frac{1}{4}(15 + 55 + 25 + 75) = 40$$

$$\begin{aligned}\text{Total Variation} &= \sum_{i=1}^4 (y_i - \bar{y})^2 \\ &= (25^2 + 15^2 + 15^2 + 35^2) \\ &= 2100 \\ &= 4 \times 20^2 + 4 \times 10^2 + 4 \times 5^2\end{aligned}$$

- Total variation= 2100

Variation due to Memory = 1600 (76%)

Variation due to cache = 400 (19%)

Variation due to interaction = 100 (5%)

# Case Study 17.1: Interconnection Nets

- ❑ Memory interconnection networks: Omega and Crossbar.
- ❑ Memory reference patterns: *Random* and *Matrix*
- ❑ Fixed factors:
  - Number of processors was fixed at 16.
  - Queued requests were not buffered but blocked.
  - Circuit switching instead of packet switching.
  - Random arbitration instead of round robin.
  - Infinite interleaving of memory  $\Rightarrow$  no memory bank contention.



## 2<sup>2</sup> Design for Interconnection Networks

Factors Used in the Interconnection Network Study

Symbol	Factor	Level	
		-1	1
A	Type of the network	Crossbar	Omega
B	Address Pattern Used	Random	Matrix

		Response		
A	B	Throughput T	90% Transit N	Response R
-1	-1	0.0641	3	1.655
1	-1	0.4220	5	2.378
-1	1	0.7922	2	1.262
1	1	0.4717	4	2.190

# Interconnection Networks Results

Parameter	Mean Estimate			Variation Explained		
	T	N	R	T	N	R
$q_0$	0.5725	3.5	1.871			
$q_A$	0.0595	-0.5	-0.145	17.2%	20%	10.9%
$q_B$	-0.1257	1.0	0.413	77.0%	80%	87.8%
$q_{AB}$	-0.0346	0.0	0.051	5.8%	0%	1.3%

- ❑ Average throughput = 0.5725
- ❑ Most effective factor = B = Reference pattern  
 $\Rightarrow$  The address patterns chosen are very different.
- ❑ Reference pattern explains  $\mp$  0.1257 (77%) of variation.
- ❑ Effect of network type = 0.0595  
 Omega networks = Average + 0.0595  
 Crossbar networks = Average - 0.0595
- ❑ Slight interaction (0.0346) between reference pattern and network type.

# General $2^k$ Factorial Designs

□  $k$  factors at two levels each.

$2^k$  experiments.

$2^k$  effects:

$k$  main effects

$\binom{k}{2}$  two factor interactions

$\binom{k}{3}$  three factor interactions...

## 2<sup>k</sup> Design Example

- Three factors in designing a machine:
  - Cache size
  - Memory size
  - Number of processors

	Factor	Level -1	Level 1
A	Memory Size	4MB	16MB
B	Cache Size	1kB	2kB
C	Number of Processors	1	2

## 2<sup>k</sup> Design Example (cont)

Cache Size	4M Bytes		16M Bytes	
	1 Proc	2 Proc	1 Proc	2 Proc
1K Byte	14	46	22	58
2K Byte	10	50	34	86

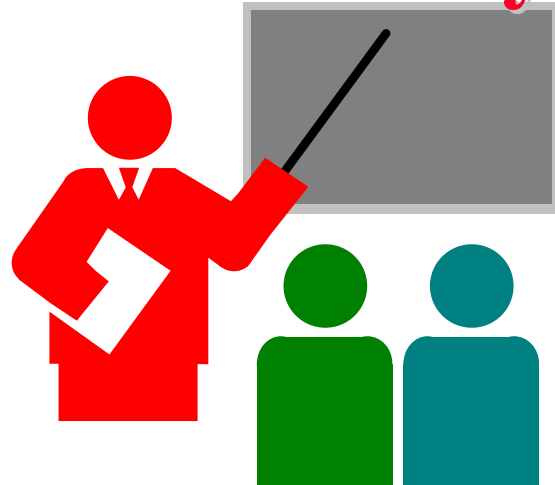
I	A	B	C	AB	AC	BC	ABC	y
1	-1	-1	-1	1	1	1	-1	14
1	1	-1	-1	-1	-1	1	1	22
1	-1	1	-1	-1	1	-1	1	10
1	1	1	-1	1	-1	-1	-1	34
1	-1	-1	1	1	-1	-1	1	46
1	1	-1	1	-1	1	-1	-1	58
1	-1	1	1	-1	-1	1	-1	50
1	1	1	1	1	1	1	1	86
320	80	40	160	40	16	24	9	Total
40	10	5	20	5	2	3	1	Total/8

## Analysis of 2<sup>k</sup> Design

$$\begin{aligned} \text{SST} &= 2^3(q_A^2 + q_B^2 + q_C^2 + q_{AB}^2 + q_{AC}^2 + q_{BC}^2 + q_{ABC}^2) \\ &= 8(10^2 + 5^2 + 20^2 + 5^2 + 2^2 + 3^2 + 1^2) \\ &= 800 + 200 + 3200 + 200 + 32 + 72 + 8 = 4512 \\ &= 18\% + 4\% + 71\% + 4\% + 1\% + 2\% + 0\% \\ &= 100\% \end{aligned}$$

- Number of Processors (C) is the most important factor.

# Summary



- ❑  $2^k$  design allows  $k$  factors to be studied at two levels each
- ❑ Can compute main effects and all multi-factors interactions
- ❑ Easy computation using sign table method
- ❑ Easy allocation of variation using squares of effects

## Exercise 17.1

Analyze the  $2^3$  design:

	$A_1$		$A_2$	
	$C_1$	$C_2$	$C_1$	$C_2$
$B_1$	100	15	120	10
$B_2$	40	30	20	50

- Quantify main effects and all interactions.
- Quantify percentages of variation explained.
- Sort the variables in the order of decreasing importance.



# Homework 17

**Modified** Exercise 17.1 Analyze the  $2^3$  design:

	$A_1$		$A_2$	
	$C_1$	$C_2$	$C_1$	$C_2$
$B_1$	110	15	120	10
$B_2$	60	30	40	50

- Quantify main effects and all interactions.
- Quantify percentages of variation explained.
- Sort the variables in the order of decreasing importance.