# Two Factors Full Factorial Design without Replications

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These slides are available on-line at:

http://www.cse.wustl.edu/~jain/cse567-06/

# Overview

- Computation of Effects
- Estimating Experimental Errors
- Allocation of Variation
- ANOVA Table
- Visual Tests
- Confidence Intervals For Effects
- Multiplicative Models
- Missing Observations

# Two Factors Full Factorial Design

- Used when there are two parameters that are carefully controlled
- Examples:
  - To compare several processors using several workloads.
  - To determining two configuration parameters, such as cache and memory sizes
- Assumes that the factors are categorical. For quantitative factors, use a regression model.
- A full factorial design with two factors *A* and *B* having *a* and *b* levels requires *ab* experiments.
- First consider the case where each experiment is conducted only once.

# Model

$$y_{ij} = \mu + \alpha_j + \beta_i + e_{ij}$$

| | | |
|---|---|---|
| $y_{ij}$ | = | Observation with A at level j and B at level i |
| $\mu$ | = | mean response |
| $\alpha_j$ | = | effect of factor A at level j |
| $\beta_i$ | = | effect of factor B at level i |
| $e_{ij}$ | = | error term |

# Computation of Effects

- Averaging the jth column produces:

$$\bar{y}_{.j} = \mu + \alpha_j + \frac{1}{b}\sum_i \beta_i + \frac{1}{b}\sum_i e_{ij}$$

- Since the last two terms are zero, we have:

$$\bar{y}_{.j} = \mu + \alpha_j$$

- Similarly, averaging along rows produces:

$$\bar{y}_{i.} = \mu + \beta_i$$

- Averaging all observations produces

$$\bar{y}_{..} = \mu$$

- Model parameters estimates are:

$$\begin{aligned} \mu &= \bar{y}_{..} \\ \alpha_j &= \bar{y}_{.j} - \bar{y}_{..} \\ \beta_i &= \bar{y}_{i.} - \bar{y}_{..} \end{aligned}$$

- Easily computed using a tabular arrangement.

# Example 21.1: Cache Comparison

| Workloads | Two Caches | One Cache | No Cache |
|---|---|---|---|
| ASM | 54.0 | 55.0 | 106.0 |
| TECO | 60.0 | 60.0 | 123.0 |
| SIEVE | 43.0 | 43.0 | 120.0 |
| DHRYSTONE | 49.0 | 52.0 | 111.0 |
| SORT | 49.0 | 50.0 | 108.0 |

# Example 21.1: Computation of Effects

| Workloads | Two Caches | One Cache | No Cache | Row Sum | Row Mean | Row Effect |
|---|---|---|---|---|---|---|
| ASM | 54.0 | 55.0 | 106.0 | 215.0 | 71.7 | -0.5 |
| TECO | 60.0 | 60.0 | 123.0 | 243.0 | 81.0 | 8.8 |
| SIEVE | 43.0 | 43.0 | 120.0 | 206.0 | 68.7 | -3.5 |
| DHRYSTONE | 49.0 | 52.0 | 111.0 | 212.0 | 70.7 | -1.5 |
| SORT | 49.0 | 50.0 | 108.0 | 207.0 | 69.0 | -3.2 |
| Column Sum | 255.0 | 260.0 | 568.0 | 1083.0 | | |
| Column Mean | 51.0 | 52.0 | 113.6 | | 72.2 | |
| Column effect | -21.2 | -20.2 | 41.4 | | | |

- An average workload on an average processor requires 72.2 ms of processor time.
- The time with two caches is 21.2 ms lower than that on an average processor
- The time with one cache is 20.2 ms lower than that on an average processor.
- The time without a cache is 41.4 ms higher than the average

# Example 21.1 (Cont)

- Two-cache - One-cache = 1 ms.
- One-cache - No-cache = 41.4-20.2 or 21.2 ms.
- The workloads also affect the processor time required.
- The ASM workload takes 0.5 ms less than the average.
- TECO takes 8.8 ms higher than the average.

# Estimating Experimental Errors

- Estimated response:

  $$\hat{y}_{ij} = \mu + \alpha_j + \beta_i$$

- Experimental error:

  $$e_{ij} = y_{ij} - \hat{y}_{ij} = y_{ij} - \mu - \alpha_j - \beta_i$$

- Sum of squared errors (SSE):

  $$\text{SSE} = \sum_{i=1}^{b}\sum_{j=1}^{a} e_{ij}^2$$

- Example: The estimated processor time is:

  $$\hat{y}_{11} = \mu + \alpha_1 + \beta_1 = 72.2 - 21.2 - 0.5 = 50.5$$

- Error = Measured-Estimated = 54-50.5 = 3.5

# Example 21.2: Error Computation

| Workloads | Two Caches | One Cache | No Cache |
|---|---|---|---|
| ASM | 3.5 | 3.5 | -7.1 |
| TECO | 0.2 | -0.8 | 0.6 |
| SIEVE | -4.5 | -5.5 | 9.9 |
| DHRYSTONE | -0.5 | 1.5 | -1.1 |
| SORT | 1.2 | 1.2 | -2.4 |

The sum of squared errors is:

$$\text{SSE} = (3.5)^2 + (0.2)^2 + \cdots + (-2.4)^2 = 2368.00$$

# Example 21.2: Allocation of Variation

$$y_{ij} = \mu + \alpha_j + \beta_i + e_{ij}$$

- Squaring the model equation:

$$\begin{array}{ccccccccc} \sum_{ij} y_{ij}^2 & = & ab\mu^2 & + & b\sum_j \alpha_j^2 & + & a\sum_i \beta_i^2 & + & \sum_{ij} e_{ij}^2 \\ \text{SSY} & = & \text{SS0} & + & \text{SSA} & + & \text{SSB} & + & \text{SSE} \end{array}$$

$$\begin{array}{ccccccccccc} \text{SST} & = & \text{SSY} & - & \text{SS0} & = & \text{SSA} & + & \text{SSB} & + & \text{SSE} \\ 13402.41 & = & 91595 & - & 78192.59 & = & 12857.20 & + & 308.40 & + & 236.80 \\ 100\% & = & & & & = & 95.9\% & + & 2.3\% & + & 1.8\% \end{array}$$

- High percent variation explained
  $\Rightarrow$ Cache choice important in processor design.

# Analysis of Variance

- Degrees of freedoms:

$$\begin{array}{ccccccccc} \text{SSY} & = & \text{SS0} & + & \text{SSA} & + & \text{SSB} & + & \text{SSE} \\ ab & = & 1 & + & (a-1) & + & (b-1) & + & (a-1)(b-1) \end{array}$$

- Mean squares:

$$\text{MSA} = \frac{\text{SSA}}{a-1}$$

$$\text{MSB} = \frac{\text{SSB}}{b-1}$$

$$\text{MSE} = \frac{\text{SSE}}{(a-1)(b-1)}$$

$$\frac{MSA}{MSE} \sim F_{[a-1,(a-1)(b-1)]}$$

- Computed ratio > $F_{[1-\alpha;a-1,(a-1)(b-1)]}$ $\Rightarrow$A is significant at level $\alpha$.

# ANOVA Table

| Component | Sum of Squares | %Variation | DF | Mean Square | $F$-Comp. | $F$-Table |
|---|---|---|---|---|---|---|
| $y$ | $\text{SSY} = \sum y_{ij}^2$ | | $ab$ | | | |
| $\bar{y}_{...}$ | $\text{SS0} = ab\mu^2$ | | 1 | | | |
| $y - \bar{y}_{...}$ | SST=SSY-SS0 | 100 | $ab-1$ | | | |
| $A$ | $\text{SSA} = b\Sigma\alpha_j^2$ | $100\left(\frac{\text{SSA}}{\text{SST}}\right)$ | $a-1$ | $\text{MSA} = \frac{\text{SSA}}{a-1}$ | $\frac{\text{MSA}}{\text{MSE}}$ | $F_{[1-\alpha, a-1, (a-1)(b-1)]}$ |
| $B$ | $\text{SSB} = a\Sigma\beta_i^2$ | $100\left(\frac{\text{SSB}}{\text{SST}}\right)$ | $b-1$ | $\text{MSB} = \frac{\text{SSB}}{b-1}$ | $\frac{\text{MSB}}{\text{MSE}}$ | $F_{[1-\alpha, b-1, (a-1)(b-1)]}$ |
| $e$ | $\text{SSE} = \text{SST} - (\text{SSA} + \text{SSB})$ | $100\left(\frac{\text{SSE}}{\text{SST}}\right)$ | $(a-1)$ $(b-1)$ | $\text{MSE} = \frac{\text{SSE}}{(a-1)(b-1)}$ | | |

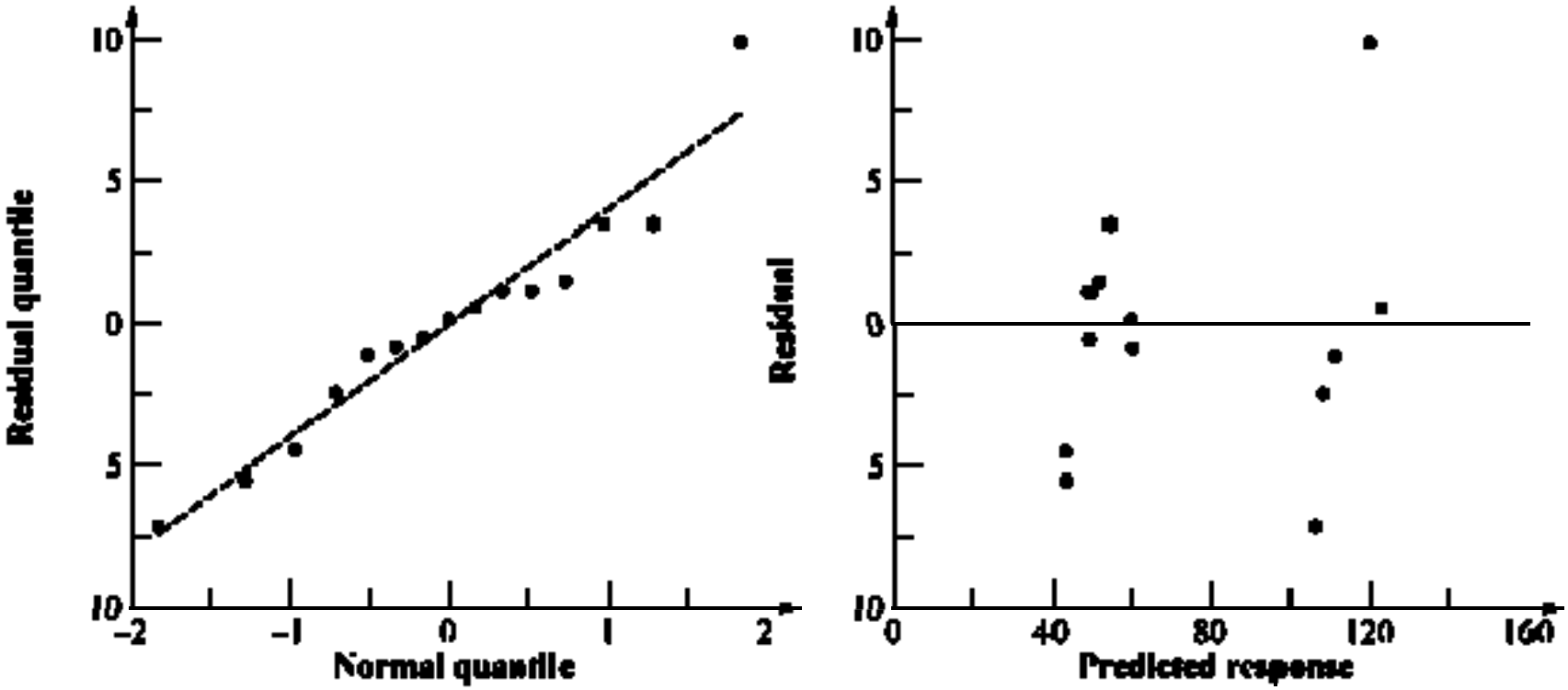
# Example 21.3: Cache Comparison

| Component | Sum of Squares | %Variation | DF | Mean Square | F-Comp. | F-Table |
|---|---|---|---|---|---|---|
| y | 91595.00 | | | | | |
| $y_{..}$ | 78192.59 | | | | | |
| y-$y_{..}$ | 13402.41 | 100.0% | 14 | | | |
| Caches | 12857.20 | 95.9% | 2 | 6428.60 | 217.2 | 3.1 |
| Workloads | 308.40 | 2.3% | 4 | 77.10 | 2.6 | 2.8 |
| Errors | 236.80 | 1.8% | 8 | 29.60 | | |

$$s_e = \sqrt{\text{MSE}} = \sqrt{29.60} = 5.44$$

- Cache choice significant.
- Workloads insignificant

# Example 21.4: Visual Tests



# Confidence Intervals For Effects

| Parameter | Estimate | Variance |
|---|---|---|
| $\mu$ | $\bar{y}_{..}$ | $s_e^2/ab$ |
| $\alpha_j$ | $\bar{y}_{.j}$-$\bar{y}_{..}$ | $s_e^2(a-1)/ab$ |
| $\mu$+$\alpha_j$ | $\bar{y}_{.j}$ | $s_e^2/b$ |
| $\beta_i$ | $\bar{y}_{i.}$-$\bar{y}_{..}$ | $s_e^2(b-1)/ab$ |
| $\mu$+$\alpha_j$+$\beta_i$ | $\bar{y}_{.j}$+$\bar{y}_{i.}$-$\bar{y}_{..}$ | $s_e^2(a+b-1)/(ab)$ |
| $\sum_{j=1}^{a} h_j\ \alpha_j, \sum_{j=1}^{a} h_j = 0$ | $\sum_{j=1}^{a} h_j\ \bar{y}_{.j}$ | $s_e^2 \sum_{j=1}^{a} h_j^2/b$ |
| $\sum_{i=1}^{b} h_i\ \beta_i, \sum_{i=1}^{b} h_i = 0$ | $\sum_{i=1}^{b} h_i\ \bar{y}_{i.}$ | $s_e^2 \sum_{i=1}^{b} h_i^2/a$ |
| $s_e^2$ | $\{\sum_{j=1}^{a} \sum_{i=1}^{b} e_{ij}^2\}/\{(a-1)(b-1)\}$ | |

Degrees of freedom for errors = (a-1)(b-1)

- For confidence intervals use *t* values at (*a*-1)(*b*-1) degrees of freedom

# Example 21.5: Cache Comparison

- Standard deviation of errors:

$$s_e = \sqrt{\text{MSE}} = \sqrt{29.60} = 5.4$$

- Standard deviation of the grand mean:

$$s_\mu = s_e/\sqrt{ab} = 5.4/\sqrt{15} = 1.4$$

- Standard deviation of $\alpha_j$'s:

$$s_{\alpha_j} = s_e\sqrt{(a-1)/ab} = 5.4\sqrt{\frac{2}{15}} = 2.8$$

- Standard deviation of $\beta_i$'s:

$$s_{\beta_i} = s_e\sqrt{(b-1)/ab} = 5.4\sqrt{\frac{4}{15}} = 2.0$$

# Example 21.5 (Cont)

- Degrees of freedom for the errors are (a-1)(b-1)=8.
  For 90% confidence interval, $t_{[0.95;8]}$= 1.86.
- Confidence interval for the grand mean:

$$72.2 \mp 1.86 \times 1.4 = 72.2 \mp 2.6 = (69.6,\ 74.8)$$

| Parameter | Mean Effect | Std. Dev. | Confidence Interval |
|---|---|---|---|
| $\mu$ | 72.2 | 1.4 | ( 69.6, 74.8) |
| Caches | | | |
| Two Caches | -21.2 | 2.8 | ( -24.9, -17.5) |
| One Cache | -20.2 | 2.8 | ( -23.9, -16.5) |
| No Cache | 41.4 | 2.8 | ( 37.7, 45.1) |

- All three cache alternatives are significantly different from the average.

# Example 21.5 (Cont)

| Parameter | Mean Effect | Std. Dev. | Confidence Interval |
|---|---|---|---|
| ASM | -0.5 | 2.0 | ( -5.8, 4.7)† |
| TECO | 8.8 | 2.0 | ( 3.6, 14.0) |
| SIEVE | -3.5 | 2.0 | ( -8.8, 1.7)† |
| DHRYSTONE | -1.5 | 2.0 | ( -6.8, 3.7)† |
| SORT | -3.2 | 2.0 | ( -8.4, 2.0)† |

† ⇒ Not significant

- All workloads, except TECO, are similar to the average and hence to each other.

# Example 21.5: CI for Differences

| | Two Caches | One Cache | No Cache |
|---|---|---|---|
| Two Caches | | ( -7.4, 5.4)† | ( -69.0, -56.2) |
| One Cache | | | ( -68.0, -55.2) |

† ⇒ Not significant

- Two-cache and one-cache alternatives are both significantly better than a no cache alternative.
- There is no significant difference between two-cache and one-cache alternatives.

# Case Study 21.1: Cache Design Alternatives

- **Multiprocess environment**: Five jobs in parallel.

  ALL = ASM, TECO, SIEVE, DHRYSTONE, and SORT in parallel.

- **Processor Time:**

| Workload | Two Caches | One Cache | No Cache |
|---|---|---|---|
| ASM5 | 231 | 262 | 489 |
| TECO5 | 300 | 314 | 620 |
| SIEVE5 | 213 | 214 | 604 |
| DHRYSTONE5 | 245 | 263 | 564 |
| ALL | 229 | 242 | 551 |

# Case Study 21.1 on Cache Design (Cont)

**Confidence Intervals for Differences:**

| | One Cache | No Cache |
|---|---|---|
| Two Caches | ( -51.6, 20.8)† | ( -358.2, -285.8) |
| One Cache | | ( -342.8, -270.4) |

† ⇒ Not significant

Conclusion: The two caches do not produce statistically better performance.

# Multiplicative Models

- Additive model:

  $$y_i = \mu + \alpha_j + \beta_i + e_{ij}$$

- If factors multiply $\Rightarrow$ Use multiplicative model
- Example: processors and workloads
  - Log of response follows an additive model
- If the spread in the residuals increases with the mean response
  $\Rightarrow$ Use transformation

# Case Study 21.2: RISC architectures

- Parallelism in time vs parallelism in space
- Pipelining vs several units in parallel
- Spectrum = HP9000/840 at 125 and 62.5 ns cycle
- Scheme86 = Designed at MIT

# Cache Study 21.2: Simulation Results

Execution Times

| Workloads | Processors | | |
|---|---|---|---|
| | Scheme86 | Spectrum125 | Spectrum62.5 |
| Garbage Collection | 39.97 | 99.06 | 56.24 |
| Pattern Match | 0.958 | 1.672 | 1.252 |
| Bignum Addition | 0.01910 | 0.03175 | 0.01844 |
| Bignum Multiplication | 0.256 | 0.423 | 0.236 |
| Fast Fourier Transform (1024) | 10.21 | 20.28 | 10.14 |

- Additive model: $\Rightarrow$ No significant difference
- Easy to see that: Scheme86 = 2 or 3 $\times$ Spectrum125
- Spectrum62.5 = 2 $\times$ Spectrum125
- Execution Time = Processor Speed $\times$ Workload Size

  $\Rightarrow$ Multiplicative model.
- Observations skewed. $y_{max}/y_{min} > 1000$

  $\Rightarrow$ Adding not appropriate

# Case Study 21.2: Multiplicative Model

- Log Transformation:

| | Processors | | | Row | Row | Row |
|---|---|---|---|---|---|---|
| Workloads | Scheme86 | Spectrum125 | Spectrum62.5 | Sum | Mean | Effect |
| Garbage Collect | 1.6017 | 1.9959 | 1.7500 | 5.3477 | 1.7826 | 1.6212 |
| Pattern Match | -0.0186 | 0.2232 | 0.0976 | 0.3022 | 0.1007 | -0.0607 |
| Bignum Add | -1.7212 | -1.4949 | -1.7447 | -4.9608 | -1.6536 | -1.8150 |
| Bignum Mult | -0.5918 | -0.3737 | -0.6271 | -1.5925 | -0.5308 | -0.6922 |
| FFT (1024) | 1.0090 | 1.3092 | 1.0060 | 3.3243 | 1.1081 | 0.9467 |
| Column Sum | 0.2791 | 1.6598 | 0.4819 | 2.4208 | | |
| Column Mean | 0.0558 | 0.3320 | 0.0964 | | 0.1614 | |
| Column effect | -0.1056 | 0.1706 | -0.0650 | | | |

- Effect of the processors is significant.
- The model explains 99.9% of variation as compared to 88% in the additive model.

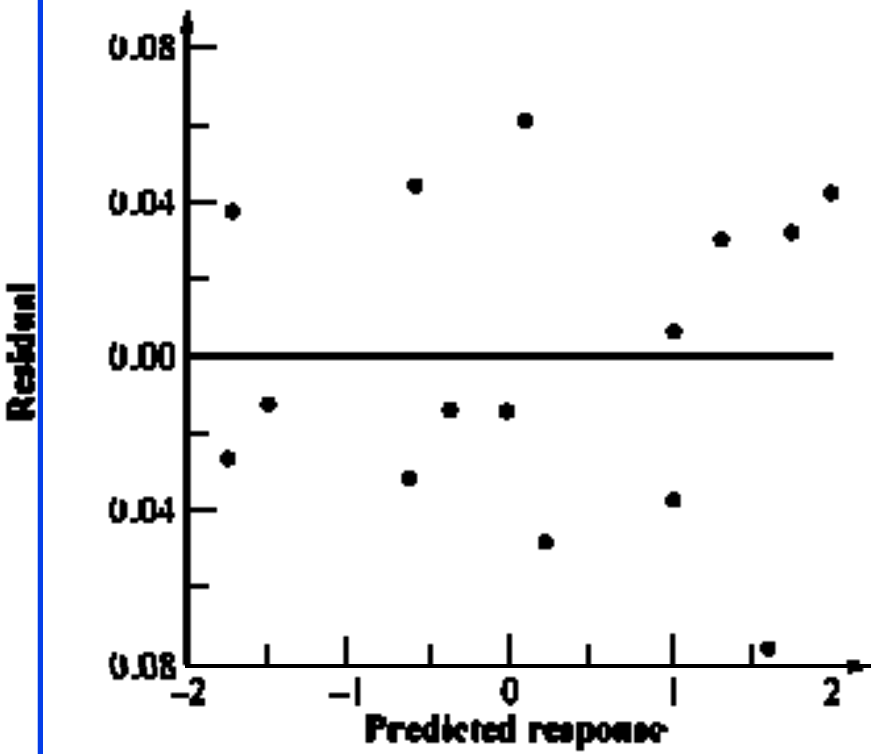
# Case Study 21.2: Confidence Intervals

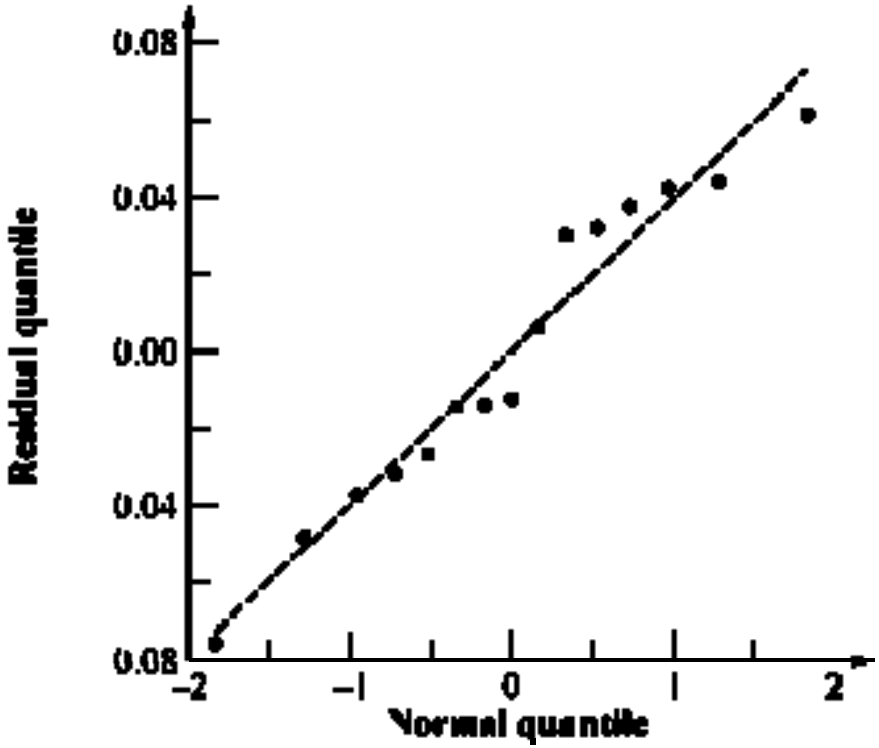
| | Spectrum125 | Spectrum62.5 |
|---|---|---|
| Scheme86 | ( -0.3387, -0.2136) | ( -0.1031, 0.0220)† |
| Spectrum125 | | ( 0.1730, 0.2982) |

† ⇒ Not significant

- Scheme86 and Spectrum62.5 are of comparable speed.
- Spectrum125 is significantly slower than the other two processors.
- Scheme86's time is 0.4584 to 0.6115 times that of Spectrum125 and 0.7886 to 1.0520 times that of Spectrum62.5.
- The time on Spectrum125 is 1.4894 to 1.9868 times that on Spectrum62.5.

# Cache Study 21.2: Visual Tests



# Case Study 21.2: ANOVA

| Component | Sum of Squares | %Variation | DF | Mean Square | F-Comp. | F-Table |
|---|---|---|---|---|---|---|
| y | 22.54 | | | | | |
| $y_{..}$ | 0.39 | | | | | |
| y-$y_{..}$ | 22.15 | 100.00% | 14 | | | |
| Processors | 0.22 | 1.00% | 2 | 0.11 | 39.29 | 3.11 |
| Workloads | 21.90 | 98.89% | 4 | 5.48 | 1935.48 | 2.81 |
| Errors | 0.02 | 0.10% | 8 | 0.00 | | |

$$s_e = \sqrt{\text{MSE}} = \sqrt{0.00} = 0.05$$

- Processors account for only 1% of the variation
- Differences in the workloads account for 99%.

  $\Rightarrow$ Workloads widely different

  $\Rightarrow$ Use more workloads or cover a smaller range.

# Case Study 21.3: Processors

Measured Elasped Times

| Work-load | Processors | | | |
|---|---|---|---|---|
| | A | B | C | D |
| ASM | 54 | 101 | 111 | 83 |
| TECO | 60 | 92 | 110 | 90 |
| SIEVE | 42 | 121 | 127 | 86 |
| DHRYSTONE | 49 | 97 | 122 | 81 |
| SORT | 52 | 100 | 107 | 82 |

# Case Study 21.3: Additive Model

| Component | Sum of Squares | %Variation | DF | Mean Square | F-Comp. | F-Table |
|---|---|---|---|---|---|---|
| y | 168653.00 | | | | | |
| $y_{..}$ | 156114.45 | | | | | |
| y-$y_{..}$ | 12538.55 | 100.0% | 19 | | | |
| Processors | 11522.15 | 91.9% | 3 | 3840.72 | 54.9 | 2.6 |
| Workloads | 176.30 | 1.4% | 4 | 44.07 | 0.6 | 2.5 |
| Errors | 840.10 | 6.7% | 12 | 70.01 | | |

- Workloads explain 1.4% of the variation.
- Only 6.7% of the variation is unexplained.

# Case Study 21.3: Multiplicative Model

| Component | Sum of Squares | %Variation | DF | Mean Square | F-Comp. | F-Table |
|---|---|---|---|---|---|---|
| y | 74.56 | | | | | |
| $y_{..}$ | 74.17 | | | | | |
| y-$y_{..}$ | 0.39 | 100.00% | 19 | | | |
| Processors | 0.36 | 93.15% | 3 | 0.121 | 57.55 | 2.61 |
| Workloads | 0.00 | 0.37% | 4 | 0.000 | 0.17 | 2.48 |
| Errors | 0.03 | 6.47% | 12 | 0.002 | | |

$$s_e = \sqrt{\text{MSE}} = \sqrt{0.002} = 0.05$$

- Both models pass the visual tests equally well.
- It is more appropriate to say that processor B takes twice as much time as processor A, than to say that processor B takes 50.7 ms more than processor A.

# Case Study 21.3: Intel iAPX 432

| System | | | | Workload | | | |
|---|---|---|---|---|---|---|---|
| No. | Processor | Language | Word Size | 484 | Sieve | Puzzle | Acker |
| 1 | VAX-11/780 | C | 32 | 1.4 | 250.0 | 9400.0 | 4600.0 |
| 2 | | Pascal (UNIX) | 32 | 1.6 | 220.0 | 11900.0 | 7800.0 |
| 3 | | Pascal (VMS) | 32 | 1.4 | 259.0 | 11530.0 | 9850.0 |
| 4 | 68000 (8 MHz) | C | 32 | 4.7 | 740.0 | 37100.0 | 7800.0 |
| 5 | | Pascal | 16 | 5.3 | 810.0 | 32470.0 | 11480.0 |
| 6 | | Pascal | 32 | 5.8 | 960.0 | 32520.0 | 12320.0 |
| 7 | 68000 (16 MHz) | Pascal | 16 | 1.3 | 196.0 | 9180.0 | 2750.0 |
| 8 | | Pascal | 32 | 1.5 | 246.0 | 9200.0 | 3080.0 |
| 9 | 8086 (5 MHz) | Pascal | 16 | 7.3 | 764.0 | 44000.0 | 11100.0 |
| 10 | 432 | Ada | 16 | 35.0 | 3200.0 | 350000.0 | 260000.0 |
| 11 | | Ada | 16 | 14.2 | 3200.0 | 165000.0 | 260000.0 |
| 12 | | Ada | 32 | 16.1 | 3200.0 | 180000.0 | 260000.0 |

# Case Study 21.3: ANOVA with Log

| Component | Sum of Squares | %Variation | DF | Mean Square | F-Comp. | F-Table |
|---|---|---|---|---|---|---|
| y | 576.64 | | | | | |
| $y_{..}$ | 449.01 | | | | | |
| y-$y_{..}$ | 127.63 | 100.0% | 47 | | | |
| Workload | 113.01 | 88.5% | 3 | 37.7 | 1158.5 | 2.3 |
| System | 13.55 | 10.6% | 11 | 1.2 | 37.9 | 1.8 |
| Errors | 1.07 | 0.8% | 33 | 0.03 | | |

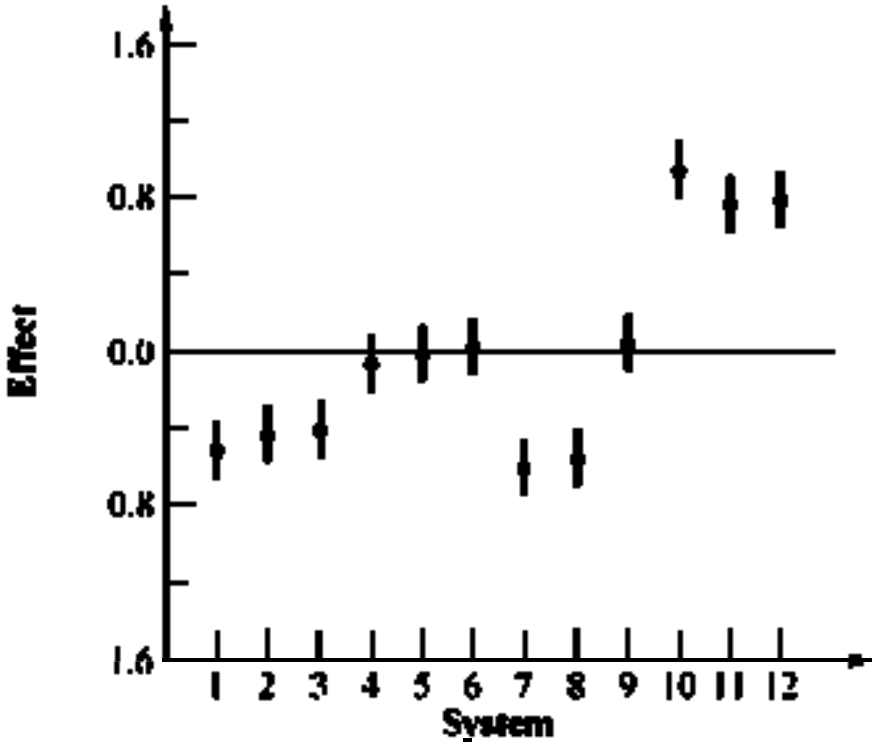
$$s_e = \sqrt{\text{MSE}} = \sqrt{0.03} = 0.18$$

- Only 0.8% of variation is unexplained.

  Workloads explain a much larger percentage of variation than the systems

  $\Rightarrow$ the workload selection is poor.

# Case Study 21.3: Confidence intervals



# Missing Observations

- Recommended Method:
  - Divide the sums by respective number of observations
  - Adjust the degrees of freedoms of sums of squares
  - Adjust formulas for standard deviations of effects
- Other Alternatives:
  - Replace the missing value by $\hat{y}$ such that the residual for the missing experiment is zero.
  - Use y such that SSE is minimum.

# Case Study 21.4: RISC-I Execution Times

| | Processors | | | | | |
|---|---|---|---|---|---|---|
| Workloads | RISC I | 68000 | Z80002 | VAX-11/780 | PDP-11/70 | C/70 |
| E-String Search | 0.46 | 1.29 | 0.74 | 0.60 | 0.41 | 1.01 |
| F-Bit Test | 0.06 | 0.29 | 0.43 | 0.29 | 0.37 | 0.55 |
| H-Linked List | 0.10 | 0.16 | 0.24 | 0.12 | 0.19 | 0.25 |
| K-Bit Matrix | 0.43 | 1.72 | 2.24 | 1.29 | 1.72 | 4.00 |
| I-Quick Sort | 50.40 | 206.64 | 262.08 | 151.20 | 181.44 | 292.32 |
| Ackermann(3,6) | 3200.00 | - | 8960.00 | 5120.00 | 5120.00 | - |
| Recursive Qsort | 800.00 | - | 4720.00 | 1840.00 | 2560.00 | 1040.00 |
| Puzzle (Subscript) | 4700.00 | - | 19740.00 | 9400.00 | 7520.00 | 15980.00 |
| Puzzle (Pointer) | 3200.00 | 13440.00 | 7360.00 | 4160.00 | 6400.00 | 6720.00 |
| SED (Batch Editor) | 5100.00 | - | 22440.00 | 5610.00 | 5610.00 | 13260.00 |
| Towers Hanoi (18) | 6800.00 | - | 28560.00 | 12240.00 | 15640.00 | 10880.00 |

# Case Study 21.5: Using Multiplicative Model

| Workloads | Processors | | | | | | Row Sum | Row Mean | Row Effect |
|---|---|---|---|---|---|---|---|---|---|
| | RISC I | 68000 | Z80002 | VAX-11/780 | PDP-11/70 | C/70 | | | |
| E-String Search | -0.34 | 0.11 | -0.13 | -0.22 | -0.38 | 0.01 | -0.96 | -0.16 | -2.16 |
| F-Bit Test | -1.22 | -0.54 | -0.36 | -0.54 | -0.43 | -0.26 | -3.36 | -0.56 | -2.55 |
| H-Linked List | -1.00 | -0.80 | -0.62 | -0.92 | -0.72 | -0.60 | -4.66 | -0.78 | -2.77 |
| K-Bit Matrix | -0.37 | 0.24 | 0.35 | 0.11 | 0.24 | 0.60 | 1.17 | 0.19 | -1.80 |
| I-Quick Sort | 1.70 | 2.32 | 2.42 | 2.18 | 2.26 | 2.47 | 13.34 | 2.22 | 0.23 |
| Ackermann(3,6) | 3.51 | - | 3.95 | 3.71 | 3.71 | - | 14.88 | 3.72 | 1.72 |
| Recursive Qsort | 2.90 | - | 3.67 | 3.26 | 3.41 | 3.02 | 16.27 | 3.25 | 1.26 |
| Puzzle (Subscript) | 3.67 | - | 4.30 | 3.97 | 3.88 | 4.20 | 20.02 | 4.00 | 2.01 |
| Puzzle (Pointer) | 3.51 | 4.13 | 3.87 | 3.62 | 3.81 | 3.83 | 22.75 | 3.79 | 1.80 |
| SED (Batch Editor) | 3.71 | - | 4.35 | 3.75 | 3.75 | 4.12 | 19.68 | 3.94 | 1.94 |
| Towers Hanoi (18) | 3.83 | - | 4.46 | 4.09 | 4.19 | 4.04 | 20.61 | 4.12 | 2.13 |
| Column Sum | 19.90 | 5.45 | 26.25 | 23.01 | 23.70 | 21.42 | 119.73 | | |
| Column Mean | 1.81 | 0.91 | 2.39 | 2.09 | 2.15 | 2.14 | | 2.00 | |
| Column effect | -0.19 | -1.09 | 0.39 | 0.10 | 0.16 | 0.15 | | | |

# Case Study 21.5: Experimental Errors

| | Processors | | | | | |
|---|---|---|---|---|---|---|
| Workloads | RISC I | 68000 | Z80002 | VAX-11/780 | PDP-11/70 | C/70 |
| E-String Search | 0.01 | 1.36 | -0.36 | -0.16 | -0.38 | 0.02 |
| F-Bit Test | -0.48 | 1.11 | -0.20 | -0.08 | -0.03 | 0.15 |
| H-Linked List | -0.04 | 1.07 | -0.23 | -0.24 | -0.10 | 0.03 |
| K-Bit Matrix | -0.37 | 1.13 | -0.24 | -0.18 | -0.12 | 0.26 |
| I-Quick Sort | -0.33 | 1.18 | -0.20 | -0.14 | -0.12 | 0.10 |
| Ackermann(3,6) | -0.03 | - | -0.16 | -0.11 | -0.17 | - |
| Recursive Qsort | -0.16 | - | 0.03 | -0.08 | - | -0.38 |
| Puzzle (Subscript) | -0.15 | - | -0.10 | -0.13 | -0.29 | 0.05 |
| Puzzle (Pointer) | -0.10 | 1.42 | -0.32 | -0.27 | -0.15 | -0.11 |
| SED (Batch Editor) | -0.04 | - | 0.02 | -0.28 | -0.35 | 0.04 |
| Towers Hanoi (18) | -0.10 | - | -0.06 | -0.13 | -0.09 | -0.23 |

# Case Study 21.5: Experimental Errors (Cont)

$$\text{SSE} = \sum_{i,j} e_{ij}^2 = 11.01$$

- 16 independent parameters ($\mu$, $\alpha_j$, and $\beta_i$) have been computed
  $\Rightarrow$ Errors have 60-1-5-10 or 44 degrees of freedom.

$$\text{MSE} = \frac{\text{SSE}}{\nu_e} = \frac{11.01}{44} = 0.25$$

- The standard deviation of errors is:

$$s_e = \sqrt{\text{MSE}} = \sqrt{0.25} = 0.05$$

- The standard deviation of $\alpha_j$:

$$s_{\alpha_j} = s_e \sqrt{\frac{N - c_j}{N c_j}}$$

- $c_j$ = number of observations in column $c_j$.

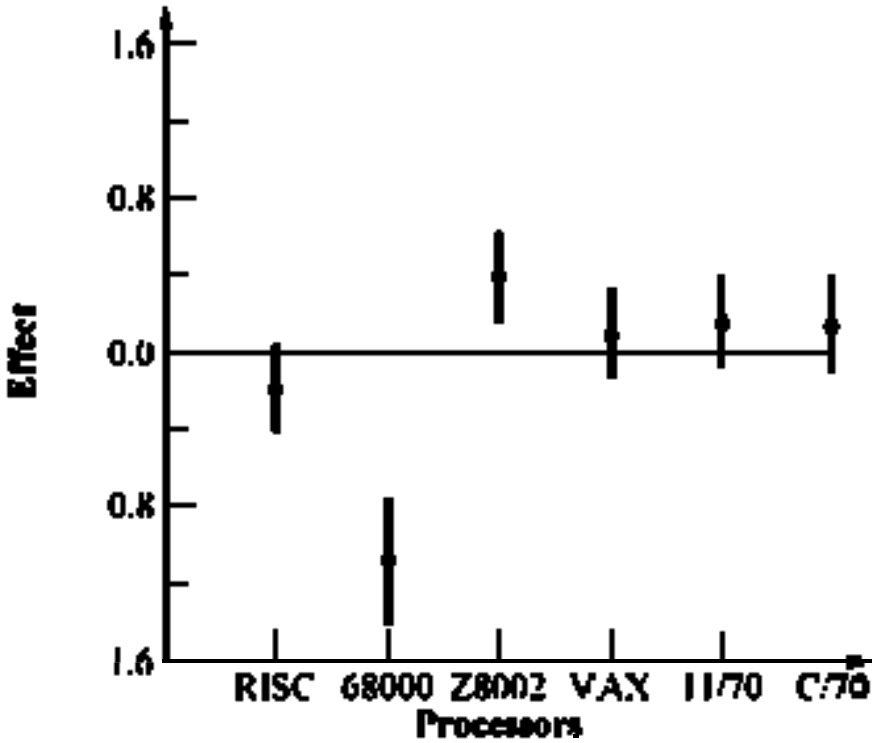
# Case Study 21.5 (Cont)

- The standard deviation of the row effects:

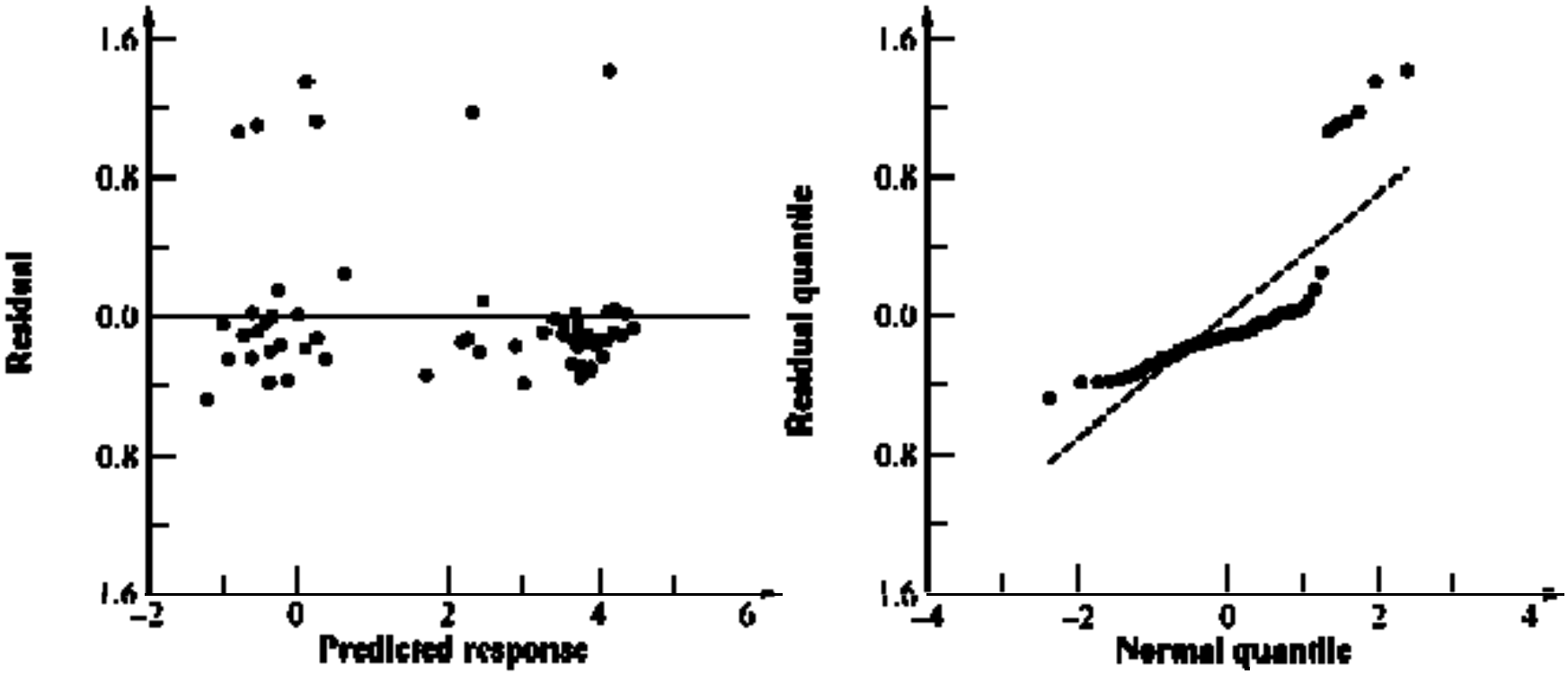
$$s_{\beta_i} = s_e\sqrt{\frac{N - r_i}{Nr_i}}$$

$r_i$=number of observations in the *i*th row.

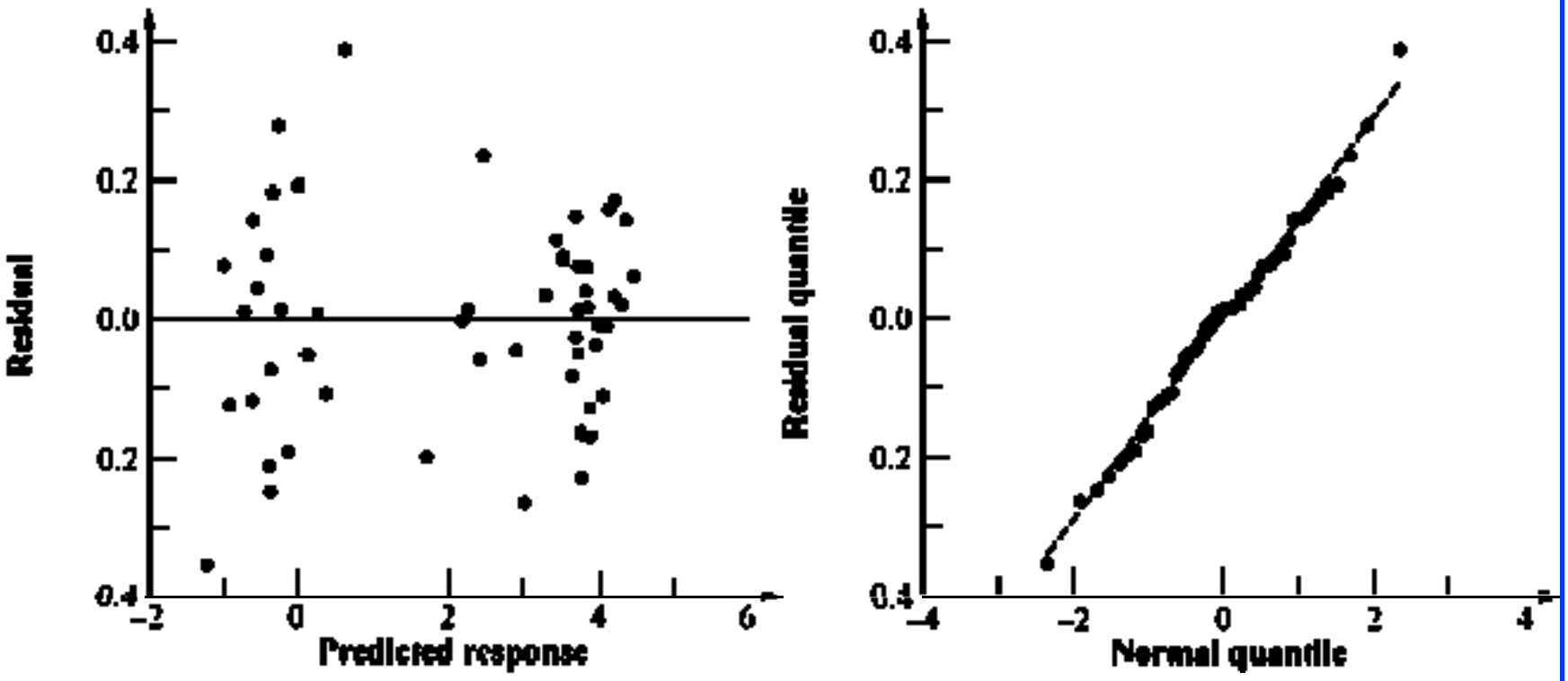
# Case Study 21.5: CIs for Processor Effects



# Case Study 21.5: Visual Tests



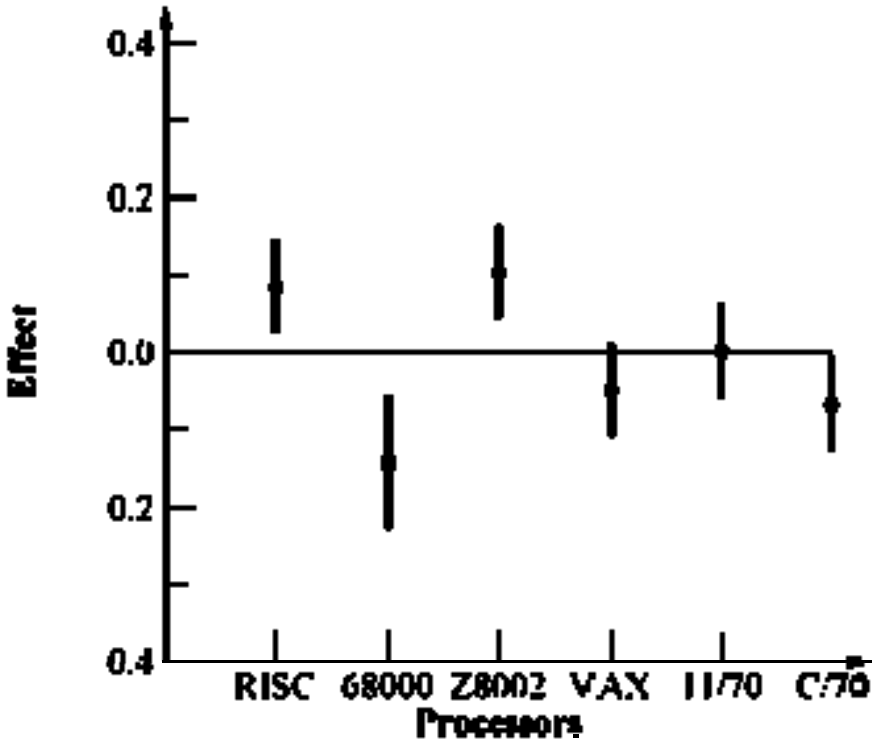
# Case Study 21.5: Analysis without 68000



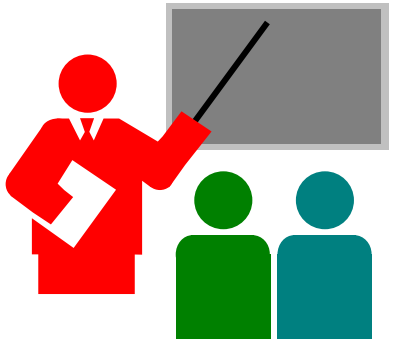
# Case Study 21.5: RISC-I Code Size

| | Processors | | | | | |
|---|---|---|---|---|---|---|
| Benchmarks | RISC I | 68000 | Z8002 | VAX-11/780 | PDP-11/70 | C/70 |
| E-String Search | 140 | 112 | 126 | 98 | 112 | 98 |
| F-Bit Test | 120 | 144 | 180 | 144 | 168 | 120 |
| H-Linked List | 176 | 123 | 141 | 211 | 299 | 141 |
| K-Bit Matrix | 288 | 317 | 374 | 288 | 374 | 317 |
| I-Quick Sort | 992 | 694 | 1091 | 893 | 1091 | 893 |
| Ackermann(3,6) | 144 | - | 302 | 72 | 86 | 86 |
| Recursive Qsort | 2736 | - | 1368 | 1368 | 1642 | 1642 |
| Puzzle (Subscript) | 2796 | 2516 | 1398 | 1398 | 1398 | 1678 |
| Puzzle (Pointer) | 752 | - | 602 | 451 | 376 | 376 |
| SED (Batch Editor) | 17720 | - | 17720 | 10632 | 8860 | 8860 |
| Towers Hanoi (18) | 96 | - | 240 | 77 | 96 | 67 |

# Case Study 21.5: Confidence Intervals



# Summary



Two Factor Designs Without Replications

- Model:

$$y_{ij} = \mu + \alpha_j + \beta_i + e_{ij}$$

- Effects are computed so that:

$$\sum_{j=1}^{a} \alpha_j = 0$$
$$\sum_{i=1}^{b} \beta_i = 0$$

- Effects:

$$\mu = \bar{y}_{..};\ \alpha_j = \bar{y}_{.j} - \bar{y}_{..};\ \beta_i = \bar{y}_{i.} - \bar{y}_{..}$$

# Summary (Cont)

- Allocation of variation: SSE can be calculated after computing

$$\begin{array}{rcccccccc} \sum_{ij} y_{ij}^2 & = & ab\mu^2 & + & b\sum_j \alpha_j^2 & + & a\sum_i \beta_i^2 & + & \sum_{ijk} e_{ijk}^2 \\ \text{SSY} & = & \text{SS0} & + & \text{SSA} & + & \text{SSB} & + & \text{SSE} \end{array}$$

Degrees of freedom:

$$\begin{array}{rcccccccc} \text{SSY} & = & \text{SS0} & + & \text{SSA} & + & \text{SSB} & + & \text{SSE} \\ ab & = & 1 & + & (a-1) & + & (b-1) & + & (a-1)(b-1) \end{array}$$

- Mean squares:

$$\text{MSA} = \frac{\text{SSA}}{a-1};\ \text{MSB} = \frac{\text{SSB}}{b-1};\ \text{MSE} = \frac{\text{SSE}}{(a-1)(b-1)}$$

- Analysis of variance:

MSA/MSE should be greater than $F_{[1-\alpha;a-1,(a-1)(b-1)]}$.
MSB/MSE should be greater than $F_{[1-\alpha;b-1,(a-1)(b-1)]}$.

# Summary (Cont)

- Standard deviation of effects:
  $s_\mu^2 = s_e^2/ab;\ s_{\alpha_j}^2 = s_e^2(a-1)/ab;\ s_{\beta_i}^2 = s_e^2(b-1)/ab;$
- Contrasts:
  For $\sum_{j=1}^{a} h_j\ \alpha_j, \sum_{j=1}^{a} h_j = 0$: Mean $= \sum_{j=1}^{a} h_j\ \bar{y}_{.j}$; Variance $= s_e^2 \sum_{j=1}^{a} h_j^2/b$
  For $\sum_{i=1}^{b} h_i\ \beta_i, \sum_{i=1}^{b} h_i = 0$: Mean $= \sum_{i=1}^{b} h_i\ \bar{y}_{i.}$; Variance $= s_e^2 \sum_{i=1}^{b} h_i^2/a$
- All confidence intervals are calculated using $t_{[1-\alpha/2;(a-1)(b-1)]}$.
- Model assumptions:
  - Errors are IID normal variates with zero mean.
  - Errors have the same variance for all factor levels.
  - The effects of various factors and errors are additive.
- Visual tests:
  - No trend in scatter plot of errors versus predicted responses
  - The normal quantile-quantile plot of errors should be linear.

# Exercise 21.1

Analyze the data of Case study 21.2 using an additive model.

- Plot residuals as a function of predicted response.
- Also, plot a normal quantile-quantile plot for the residuals.
- Determine 90% confidence intervals for the paired differences.
- Are the processors significantly different?
- Discuss what indicators in the data, analysis, or plot would suggest that this is not a good model.

# Exercise 21.2

Analyze the data of Table 21.18 using a multiplicative model and verify your analysis with the results presented in Table 21.19.

# Exercise 21.3

Analyze the code size data of Table 21.23. Ignore the second column corresponding to 68000 for this exercise.

Answer the following:

a. What percentage of variation is explained by the processor?

b. What percentage of variation can be attributed to the workload?

c. Is there a significant (at 90% confidence) difference between any two processors?

# Exercise 21.4

Repeat Exercise 21.3 with the 68000 column included.

# Homework

- Submit answer to Exercise 21.1