One Factor Experiments

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These slides are available on-line at:

http://www.cse.wustl.edu/~jain/cse567-06/

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- Computation of Effects
- Estimating Experimental Errors
- Allocation of Variation
- ANOVA Table and F-Test
- Visual Diagnostic Tests
- Confidence Intervals For Effects
- Unequal Sample Sizes

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One Factor Experiments

- Used to compare alternatives of a single categorical variable.

$$y_{ij} = \mu + \alpha_j + e_{ij}$$

For example, several processors, several caching schemes

$$\begin{aligned} \text{r} &= \text{Number of replications} \\ y_{ij} &= \text{ith response with jth alternative} \\ \mu &= \text{mean response} \\ \alpha_j &= \text{Effect of alternative j} \\ e_{ij} &= \text{Error term} \end{aligned}$$

$$\sum \alpha_j = 0$$

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Computation of Effects

$$\sum_{i=1}^{r}\sum_{j=1}^{a} y_{ij} = ar\mu + r\sum_{j=1}^{a}\alpha_j + \sum_{i=1}^{r}\sum_{j=1}^{a} e_{ij}$$

$$= ar\mu + 0 + 0$$

$$\mu = \frac{1}{ar}\sum_{i=1}^{r}\sum_{j=1}^{a} y_{ij} = \bar{y}_{..}$$

Computation of Effects (Cont)

$$\begin{aligned}
\bar{y}_{.j} &= \frac{1}{r}\sum_{i=1}^{r} y_{ij} \\
&= \frac{1}{r}\sum_{i=1}^{r}(\mu + \alpha_j + e_{ij}) \\
&= \frac{1}{r}\left(r\mu + r\alpha_j + \sum_{i=1}^{r} e_{ij}\right) \\
&= \mu + \alpha_j + 0
\end{aligned}$$

$$\alpha_j = \bar{y}_{.j} - \mu = \bar{y}_{.j} - \bar{y}_{..}$$

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Example 20.1: Code Size Comparison

R	V	Z
144	101	130
120	144	180
176	211	141
288	288	374
144	72	302

- Entries in a row are unrelated.
  (Otherwise, need a two factor analysis.)

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Example 20.1 Code Size (Cont)

	R	V	Z	
	144	101	130	
	120	144	180	
	176	211	141	
	288	288	374	
	144	72	302	
Col Sum	$\Sigma\ y_{.1}$ =872	$\Sigma\ y_{.2}$ =816	$\Sigma\ y_{.3}$ =1127	$\Sigma\ y_{..}$ = 2815
Col Mean	$\bar{y}_{.1}$ =174.4	$\bar{y}_{.2}$ =163.2	$\bar{y}_{.3}$ =225.4	$\mu=\bar{y}_{..}$=187.7
Col Effect	$\alpha_1=\bar{y}_{.1}-\bar{y}_{..}$ =-13.3	$\alpha_2=\bar{y}_{.2}-\bar{y}_{..}$ =-24.5	$\alpha_3=\bar{y}_{.3}-\bar{y}_{..}$ =37.7	

Example 20.1: Interpretation

- Average processor requires 187.7 bytes of storage.
- The effects of the processors R, V, and Z are -13.3, -24.5, and 37.7, respectively. That is,
  - R requires 13.3 bytes less than an average processor
  - V requires 24.5 bytes less than an average processor, and
  - Z requires 37.7 bytes more than an average processor.

Estimating Experimental Errors

- Estimated response for *j*th alternative:

$$\hat{y}_j = \mu + \alpha_j$$

- Error:

$$e_{ij} = y_j - \hat{y}_j$$

- Sum of squared errors (SSE):

$$\mathrm{SSE} = \sum_{i=1}^{r}\sum_{j=1}^{a} e_{ij}^2$$

Example 20.2

$$
\begin{bmatrix} 144 & 101 & 130 \\ 120 & 144 & 180 \\ 176 & 211 & 141 \\ 288 & 288 & 374 \\ 144 & 72 & 302 \end{bmatrix} = \begin{bmatrix} 187.7 & 187.7 & 187.7 \\ 187.7 & 187.7 & 187.7 \\ 187.7 & 187.7 & 187.7 \\ 187.7 & 187.7 & 187.7 \\ 187.7 & 187.7 & 187.7 \end{bmatrix}
$$

$$
+ \begin{bmatrix} -13.3 & -24.5 & 37.7 \\ -13.3 & -24.5 & 37.7 \\ -13.3 & -24.5 & 37.7 \\ -13.3 & -24.5 & 37.7 \\ -13.3 & -24.5 & 37.7 \end{bmatrix}
$$

$$
+ \begin{bmatrix} -30.4 & -62.2 & -95.4 \\ -54.4 & -19.2 & -45.4 \\ 1.6 & 47.8 & -84.4 \\ 113.6 & 124.8 & 148.6 \\ -30.4 & -91.2 & 76.6 \end{bmatrix}
$$

$$
\text{SSE} = (-30.4)^2 + (-54.4)^2 + \cdots + (76.6)^2 = 94365.20
$$

Allocation of Variation

$$y_{ij} = \mu + \alpha_j + e_{ij}$$

$$y_{ij}^2 = \mu^2 + \alpha_j^2 + e_{ij}^2 + 2\mu\alpha_j + 2\mu e_{ij} + 2\alpha_j e_{ij}$$

$$\sum_{i,j} y_{ij}^2 \quad = \quad \sum_{i,j} \mu^2 + \sum_{i,j} \alpha_j^2 + \sum_{i,j} e_{ij}^2$$
$$+\text{Cross product terms}$$

$$\text{SSY} = \text{SS0} + \text{SSA} + \text{SSE}$$

$$\text{SS0} = \sum_{i=1}^{r} \sum_{j=1}^{a} \mu^2 = ar\mu^2$$

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Allocation of Variation (Cont)

$$\text{SSA} = \sum_{i=1}^{r}\sum_{j=1}^{a}\alpha_j^2 = r\sum_{j=1}^{a}\alpha_j^2$$

- Total variation of y (SST):

$$\text{SST} = \sum_{i,j}(y_{i,j} - \bar{y}_{..})^2 = \sum_{i,j} y_{ij}^2 - ar\bar{y}_{..}^2 = \text{SSY} - \text{SS0} = \text{SSA} + \text{SSE}$$

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Example 20.3

$$
\begin{aligned}
\text{SSY} &= 144^2 + 120^2 + \cdots + 302^2 = 633639 \\
\text{SS0} &= ar\mu^2 \\
&= 3 \times 5 \times (187.7)^2 = 528281.7 \\
\text{SSA} &= r \sum_j \alpha_j^2 \\
&= 5[(-13.3)^2 + (-24.5)^2 + (37.6)^2] \\
&= 10992.1 \\
\text{SST} &= \text{SSY} - \text{SS0} \\
&= 633639.0 - 528281.7 = 105357.3 \\
\text{SSE} &= \text{SST} - \text{SSA} \\
&= 105357.3 - 10992.1 = 94365.2
\end{aligned}
$$

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Example 20.3 (Cont)

$$\text{Percent variation explained by processors} = 100 \times \frac{10992.13}{105357.3} = 10.4\%$$

- 89.6% of variation in code size is due to experimental errors (programmer differences).

  Is 10.4% statistically significant?

Analysis of Variance (ANOVA)

- Importance $\neq$ Significance
- Important $\Rightarrow$ Explains a high percent of variation
- Significance
  $\Rightarrow$ High contribution to the variation compared to that by errors.
- Degree of freedom
  = Number of independent values required to compute

$$\begin{array}{ccccccccc} \text{SSY} & = & \text{SS0} & + & \text{SSA} & + & \text{SSE} \\ \text{ar} & = & 1 & + & (\text{a-1}) & + & \text{a(r-1)} \end{array}$$

Note that the degrees of freedom also add up.

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F-Test

- Purpose: To check if SSA is *significantly* greater than SSE.

  Errors are normally distributed $\Rightarrow$ SSE and SSA have chi-square distributions.

  The ratio $(SSA/\nu_A)/(SSE/\nu_e)$ has an F distribution.

  where ν_A=a-1 = degrees of freedom for SSA

  ν_e=a(r-1) = degrees of freedom for SSE

  Computed ratio > $F_{[1-\alpha;\, \nu_A,\, \nu_e]}$
  $\Rightarrow$ SSA is significantly higher than SSE.

  SSA/ν_A is called mean square of A or (MSA).

  Similary, $MSE=SSE/\nu_e$

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ANOVA Table for One Factor Experiments

Component	Sum of Squares	%Variation	DF	Mean Square	F-Comp.	F-Table
y	SSY=$\sum y_{ij}^2$		ar			
$\bar{y}_{..}$	SS0=$ar\mu^2$		1			
y-$\bar{y}_{..}$	SST=SSY-SS0	100	ar-1			
A	SSA $= r\Sigma\ \alpha_i^2$	$100\left(\frac{\text{SSA}}{\text{SST}}\right)$	a-1	$\text{MSA} = \frac{\text{SSA}}{a-1}$	$\frac{\text{MSA}}{\text{MSE}}$	$F_{[1-\alpha; a-1, a(r-1)]}$
e	SSE=SST- SSA	$100\left(\frac{\text{SSE}}{\text{SST}}\right)$	$a(r-1)$	$\text{MSE} = \frac{\text{SSE}}{a(r-1)}$		

Example 20.4: Code Size Comparison

Component	Sum of Squares	%Variation	DF	Mean Square	F-Comp.	F-Table
y	633639.00					
$y_{..}$	528281.69					
y-$y_{..}$	105357.31	100.0%	14			
A	10992.13	10.4%	2	5496.1	0.7	2.8
Errors	94365.20	89.6%	12	7863.8		

$$s_e = \sqrt{\text{MSE}} = \sqrt{7863.77} = 88.68$$

- Computed F-value < F from Table

  $\Rightarrow$ The variation in the code sizes is mostly due to experimental errors and not because of any significant difference among the processors.

Visual Diagnostic Tests

Assumptions:

1. Factors effects are additive.
2. Errors are additive.
3. Errors are independent of factor levels.
4. Errors are normally distributed.
5. Errors have the same variance for all factor levels.

Tests:

- Residuals versus predicted response:
  No trend $\Rightarrow$ Independence
  Scale of errors << Scale of response
  $\Rightarrow$ Ignore visible trends.
- Normal quantilte-quantile plot linear $\Rightarrow$ Normality

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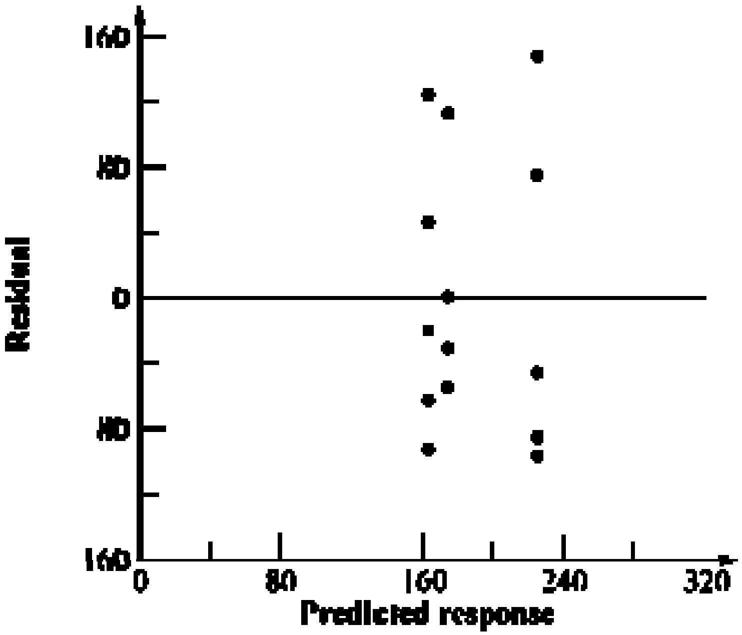
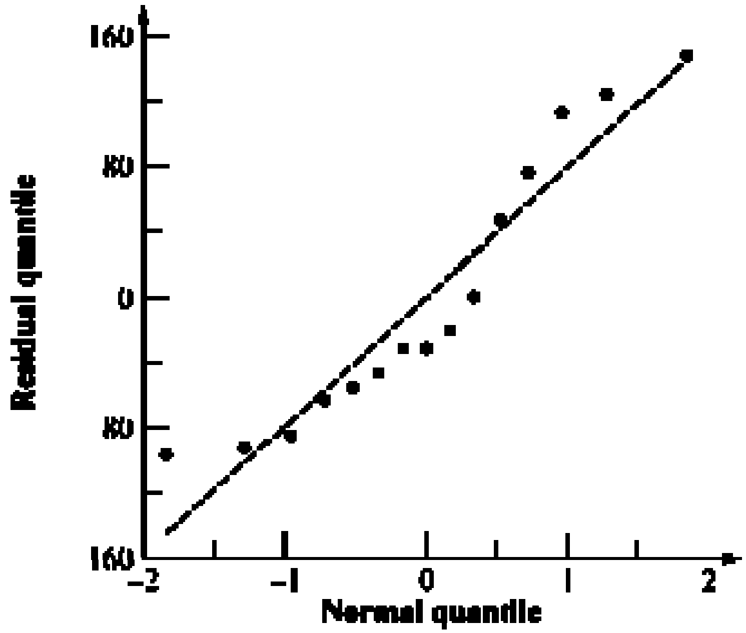
Example 20.5





- Horizontal and vertical scales similar
  ⇒ Residuals are not small ⇒ Variation due to factors is small compared to the unexplained variation
- No visible trend in the spread
- Q-Q plot is S-shaped ⇒ shorter tails than normal.

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Confidence Intervals For Effects

- Estimates are random variables

Parameter	Estimate	Variance
μ	$\bar{y}_{..}$	s_e^2/ar
α_j	$\bar{y}_{.j}$-$\bar{y}_{..}$	$s_e^2(a-1)/ar$
$\mu+\alpha_j$	$\bar{y}_{.j}$	s_e^2/r
$\sum_{j=1}^a h_j\alpha_j$, $\sum_{j=1}^a h_j=0$	$\sum_{j=1}^a h_j\ \bar{y}_{.j}$	$\sum_{j=1}^a s_e^2\ h_j^2/r$
s_e^2	$\frac{\sum e_{ij}^2}{a(r-1)}$	

Degrees of freedom for errors = a(r-1)

- For the confidence intervals, use t values at a(r-1) degrees of freedom.
- Mean responses: $\hat{y}_j = \mu + \alpha_j$
- Contrasts $\sum h_j \alpha_j$: Use for α_1-α_2

Example 20.6: Code Size Comparison

Error variance $s_e^2 = \frac{94365.2}{12} = 7863.8$

$$\begin{aligned}\text{Std Dev of errors} &= \sqrt{(\text{Var. of errors})}\\ &= 88.7\end{aligned}$$

$$\text{Std Dev of } \mu = s_e/\sqrt{ar} = 88.7/\sqrt{15} = 22.9$$

$$\begin{aligned}\text{Std Dev of } \alpha_j &= s_e\sqrt{\{(a-1)/(ar)\}}\\ &= 88.7\sqrt{(2/15)} = 32.4\end{aligned}$$

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Example 20.6 (Cont)

- For 90% confidence, $t_{[0.95;\,12]} = 1.782$.
- 90% confidence intervals:

$$
\begin{aligned}
\mu &= 197.7 \mp (1.782)(22.9) = (146.9, 228.5) \\
\alpha_1 &= -13.3 \mp (1.782)(32.4) = (-71.0, 44.4) \\
\alpha_2 &= -24.5 \mp (1.782)(32.4) = (-82.2, 33.2) \\
\alpha_3 &= 37.6 \mp (1.782)(32.4) = (-20.0, 95.4)
\end{aligned}
$$

- The code size on an average processor is significantly different from zero.
- Processor effects are not significant.

Example 20.6 (Cont)

- Using h_1=1, h_2=-1, h_3=0, ($\sum h_j$=0):

$$\text{Mean } \alpha_1 - \alpha_2 = \bar{y}_{.1} - \bar{y}_{.2} = 174.4 - 163.2 = 11.2$$

$$\begin{aligned} \text{Std dev of } \alpha_1 - \alpha_2 &= s_e\sqrt{(\sum h_j^2/r)} \\ &= 88.7\sqrt{(2/5)} = 56.1 \end{aligned}$$

$$\begin{aligned} 90\% \text{ CI for } \alpha_1 - \alpha_2 &= 11.2 \mp (1.782)(56.1) \\ &= (-88.7,\ 111.1) \end{aligned}$$

- CI includes zero ⇒ one isn't superior to other.

Example 20.6 (Cont)

- Similarly,

$$90\% \text{ CI for } \alpha_1 - \alpha_3$$
$$= (174.4 - 225.4) \mp (1.782)(56.1)$$
$$= (-150.9,\ 48.9)$$

$$90\% \text{ CI for } \alpha_2 - \alpha_3$$
$$= (163.2 - 225.4) \mp (1.782)(56.1)$$
$$= (-162.1,\ 37.7)$$

- Any one processor is not superior to another.

Unequal Sample Sizes

$$y_{ij} = \mu + \alpha_j + e_{ij}$$

- By definition:

$$\sum_{j=1}^{a} r_j \alpha_j = 0$$

- Here, r_j is the number of observations at *j*th level. N =total number of observations:

$$N = \sum_{j=1}^{a} r_j$$

Parameter Estimation

Parameter	Estimate	Variance
μ	$\bar{y}_{..}$	s_e^2/N
α_j	$\bar{y}_{.j}$-$\bar{y}_{..}$	$s_e^2(N-r_j)/(Nr_j)$
μ+α_j	$\bar{y}_{.j}$	s_e^2/r_j
$\Sigma\ h_j\alpha_j,\ \Sigma\ h_j$=0	$h_j\ \bar{y}_{.j}$	$s_e^2\sum_{j=1}^{a}\ (h_j^2/r_j)$
s_e^2	$\sum\ e_{ij}^2/\{N-a\}$	

Degrees of freedom for errors = N-a

Analysis of Variance

Component	Sum of Squares	%Variation	DF	Mean Square	F-Comp.	F-Table
y	SSY=$\sum y_{ij}^2$		N			
$\bar{y}_{..}$	SS0=$N\mu^2$		1			
y-$\bar{y}_{..}$	SST=SSY-SS0	100	N-1			
A	SSA $= \sum_{j=1}^{a} r_j\alpha_j^2$	$100\left(\frac{\text{SSA}}{\text{SST}}\right)$	a-1	MSA=$\frac{\text{SSA}}{a-1}$	$\frac{\text{MSA}}{\text{MSE}}$	$F_{[1-\alpha;a-1,N-a]}$
e	SSE=SST- SSA	$100\left(\frac{\text{SSE}}{\text{SST}}\right)$	N-a	MSE=$\frac{\text{SSE}}{N-a}$		

Example 20.7: Code Size Comparison

	R	V	Z		
	144	101	130		
	120	144	180		
	176	211	141		
	288	288			
	144				
Column Sum	872	744	451	2067	
Column Mean	174.40	186.00	150.33		172.25
Column effect	2.15	13.75	-21.92		

- All means are obtained by dividing by the number of observations added.
- The column effects are 2.15, 13.75, and -21.92.

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Example 20.6: Analysis of Variance

$$\begin{bmatrix} 144 & 101 & 130 \\ 120 & 144 & 180 \\ 176 & 211 & 141 \\ 288 & 288 & \\ 144 & & \end{bmatrix} = \begin{bmatrix} 172.25 & 172.25 & 172.25 \\ 172.25 & 172.25 & 172.25 \\ 172.25 & 172.25 & 172.25 \\ 172.25 & 172.25 & \\ 172.25 & & \end{bmatrix} + \begin{bmatrix} 2.15 & 13.75 & -21.92 \\ 2.15 & 13.75 & -21.92 \\ 2.15 & 13.75 & -21.92 \\ 2.15 & 13.75 & \\ 2.15 & & \end{bmatrix}$$

$$+ \begin{bmatrix} -30.40 & -85.00 & -20.33 \\ -54.40 & -42.00 & 29.67 \\ 1.60 & 25.00 & -9.33 \\ 113.60 & 102.00 & \\ -30.40 & & \end{bmatrix}$$

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Example 20.6 ANOVA (Cont)

- Sums of Squares:

$$
\begin{aligned}
\text{SSY} &= \sum y_{ij}^2 = 397375 \\
\text{SS0} &= N\mu^2 = 356040.75 \\
\text{SSA} &= 5\alpha_1^2 + 4\alpha_2^2 \\
&\quad +3\alpha_3^2 = 2220.38 \\
\text{SSE} &= (-30.40)^2 + (-54.40)^2 + \cdots \\
&\quad +(-9.33)^2 = 39113.87 \\
\text{SST} &= \text{SSY} - \text{SS0} = 41334.25
\end{aligned}
$$

- Degrees of Freedom:

$$
\begin{array}{ccccccccc}
\text{SSY} & = & \text{SS0} & + & \text{SSA} & + & \text{SSE} \\
\text{N} & = & 1 & + & \text{(a-1)} & + & \text{N-a} \\
12 & = & 1 & + & 2 & + & 9
\end{array}
$$

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Example 20.6 ANOVA Table

Component	Sum of Squares	%Variation	DF	Mean Square	F-Comp.	F-Table
y	397375.00					
$y_{..}$	356040.75					
y-$y_{..}$	41334.25	100.00%	11			
A	2220.38	5.37%	2	1110.19	0.26	3.01
Errors	39113.87	94.63%	9	4345.99		

$$s_e=\sqrt{\text{MSE}}=\sqrt{4345.99}=65.92$$

- **Conclusion**: Variation due processors is insignificant as compared to that due to modeling errors.

Example 20.6 Standard Dev. of Effects

- Consider the effect of processor Z: Since,

$$\begin{aligned}
\alpha_3 &= y_{.3} - y_{..} \\
&= \frac{1}{3}(y_{13} + y_{23} + y_{33}) - \frac{1}{12}(y_{11} \\
&\quad + y_{21} + \cdots + y_{32} + y_{42} + y_{13} + y_{23} + y_{33}) \\
&= \frac{1}{4}(y_{13} + y_{23} + y_{33}) - \frac{1}{12}(y_{11} + y_{21} + \cdots + y_{32} + y_{42})
\end{aligned}$$

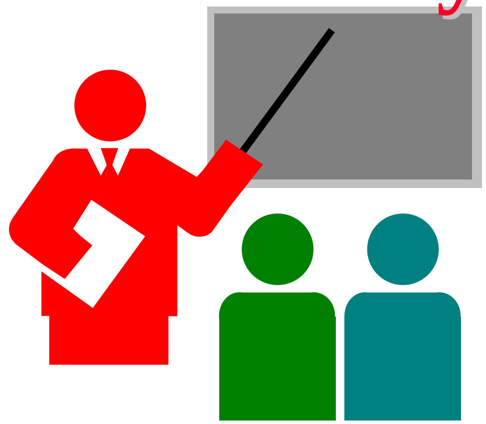
- Error in $\alpha_3 = \sum$ Errors in terms on the right hand side:

$$e_{\alpha_3} = \frac{1}{4}(e_{13} + e_{23} + e_{33}) - \frac{1}{12}(e_{11} + e_{21} + \cdots + e_{32} + e_{42})$$

- e_{ij}'s are normally distributed $\Rightarrow \alpha_3$ is normal with

$$s^2_{\alpha_3} = \frac{1}{4^2} \times 3s_e^2 + \frac{1}{12^2} \times 9s_e^2 = 1086.36$$

Summary



- Model for One factor experiments:

$$y_{ij} = \mu + \alpha_j + e_{ij} \qquad \sum_{j=1}^{a} \alpha_j = 0$$

- Computation of effects
- Allocation of variation, degrees of freedom
- ANOVA table
- Standard deviation of errors
- Confidence intervals for effects and contracts
- Model assumptions and visual tests

Exercise 20.1

For a single factor design, suppose we want to write an expression for α_j in terms of y_{ij}'s:

$$\alpha_j = a_{11j} y_{11} + a_{12j} y_{12} + \cdots + a_{raj} y_{ra}$$

What are the values of $a_{..j}$'s? From the above expression, the error in α_j is seen to be:

$$e_{\alpha_j} = a_{11j} e_{11} + a_{12j} e_{12} + \cdots + a_{raj} e_{ra}$$

Assuming errors e_{ij} are normally distributed with zero mean and variance σ_e^2, write an expression for variance of e_{α_j}. Verify that your answer matches that in Table 20.5.

Homework

Analyze the following one factor experiment:

R	V	Z
145	102	131
120	144	180
177	212	142
288		
144		

1. Compute the effects
2. Prepare ANOVA table
3. Compute confidence intervals for effects and interpret
4. Compute Confidence interval for α_1-α_3
5. Show graphs for visual tests and interpret