A Compact, Closed Form Solution for the Optimum, Ideal Wind Turbines

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References


Rotor Inflow Geometry
Background

\[ u = aU, \quad v = a'\Omega r \]

\[ \frac{a(1-a)}{a'(1+a')} = \lambda_r^2 \]
Background

\[dC_p = 8a'(1 - a)\left(\frac{\lambda_r^3}{\lambda^2}\right)\,d\lambda_r\]

\[\lambda_r^2 = \frac{(1 - a)(4a - 1)^2}{(1 - 3a)}, \quad a' = \frac{(1 - 3a)}{(4a - 1)}\]

\[16a^3 - 24a^2 + (9 - 3\lambda_r^2)a + \lambda_r^2 - 1 = 0\]
Background

\[ \phi = \tan^{-1}\left(\frac{1-a}{\lambda_r (1+a')}\right) = \tan^{-1}\left(\lambda_r \frac{a'}{a}\right) \]
Background

\[ C_p = \frac{8}{729\lambda^2} \left\{ \frac{64}{5} x^5 + 72x^4 + 124x^3 + 38x^2 - 63x - 12\left[\ln(x)\right] - \frac{4}{x} \right\}_{x=0.25}^{x=1-3a_0} \]
Alternative Approach

\[ V^2 = U^2 + (\Omega r)^2 - w^2 \]

\[ \sin(\phi) = \frac{U \sqrt{U^2 + (\Omega r)^2 - w^2 - \Omega rw}}{U^2 + (\Omega r)^2} \]

\[ \cos(\phi) = \frac{(\Omega r) \sqrt{U^2 + (\Omega r)^2 - w^2 + Uw}}{U^2 + (\Omega r)^2} \]
Alternate Approach

$$\eta^2 = 1 + \lambda_r^2 - b^2$$

$$\sin(\phi) = \frac{\sqrt{1 + \lambda_r^2 - b^2} - \lambda_r b}{1 + \lambda_r^2} = \frac{v}{w}$$

$$\cos(\phi) = \frac{\lambda_r \sqrt{1 + \lambda_r^2 - b^2} + b}{1 + \lambda_r^2} = \frac{u}{w}$$
Momentum Theory

\[ dL = 2\rho(2\pi rdr)w[U - w\cos(\phi)] \]

\[ dP = 2\rho(2\pi rdr)w[U - w\cos(\phi)](\Omega r)\sin(\phi) \]

\[ dC_p = 8\lambda_r b[1 - b\cos(\phi)]\sin(\phi)\overline{rdr} \]

\[ 16z^3 - 24z^2 + 9z - \frac{1}{(1 + \lambda_r^2)} = 0 \]

Where:

\[ z = b^2/(1 + \lambda_r^2) \]
Momentum Theory

\[ b^2 = \left(\frac{1 + \lambda_r^2}{2}\right) \left[ 1 + \cos \left( \frac{2\pi}{3} + \frac{1}{3} \cos^{-1} \left( \frac{1 - \lambda_r^2}{1 + \lambda_r^2} \right) \right) \right] \]

\[ \frac{1}{b} = 1 + 2 \cos \left[ \frac{\pi}{3} - \frac{1}{3} \cos^{-1} \left( \frac{1 - \lambda_r^2}{1 + \lambda_r^2} \right) \right] \]

\[ \frac{1}{b} = 1 + \cos \left[ \frac{1}{3} \cos^{-1} \left( \frac{1 - \lambda_r^2}{1 + \lambda_r^2} \right) \right] + \sqrt{3} \sin \left[ \frac{1}{3} \cos^{-1} \left( \frac{1 - \lambda_r^2}{1 + \lambda_r^2} \right) \right] \]

The axial and swirl induction factors:

\[ a = b \cos(\phi) \]

\[ f = b \sin(\phi) \]
Complete Expressions

\[ \lambda_r = \frac{\sqrt{1 + b (1 - 2b)}}{\sqrt{3b - 1}} \]

\[ b = \frac{1}{1 + 2\cos(\phi)} \]
Complete Expressions

\[ \sin(\phi) = \frac{\sqrt{1+b} \sqrt{3b} - 1}{2b} \]

\[ \cos(\phi) = \frac{1 - b}{2b} \]
Complete Expressions

\[ \phi = \frac{\pi}{3} - \frac{1}{3} \cos^{-1} \left( \frac{1 - \lambda_r^2}{1 + \lambda_r^2} \right) \]

\[ \lambda_r = \frac{\cos^2(\phi) - \sin^2(\phi) + \cos(\phi)}{\sin(\phi)(1 + 2\cos(\phi))} = \frac{1}{\tan\left(\frac{3\phi}{2}\right)} \]
Complete Expressions

\[ a = b \cos(\phi) = \frac{1-b}{2} = \frac{\cos(\phi)}{1 + 2 \cos(\phi)} \]

\[ b = 1 - 2a \]

\[ a' = \frac{f}{\lambda_r} = \frac{b \sin(\phi)}{\lambda_r} = \frac{\sin(\phi)}{\lambda_r (1 + 2 \cos(\phi))} \]

Total flow at the blade in terms of \( b \):

\[ \eta^2 = 1 + \lambda_r^2 - b^2 = \frac{b^2 (1+b)}{3b-1} \]
Wake Induction Parameters as a Function of Local Speed Ratio

Figure 2. Wake Induction Parameters as a Function of Local Speed Ratio
Optimized Inflow Angle as a Function of Local Speed Ratio

Figure 3. Optimum Inflow Angle as a Function of Local Speed Ratio
Optimized Inflow Angle as a Function of $\sin^2$ of Initial Inflow Ratio

Figure 4. Optimized Inflow Angle as a Function of $\sin^2$ of Initial Inflow Ratio
Optimum Power Coefficient

\[ dC_p = 2(1+b)^2(1-2b)\bar{r}dr = \frac{2(1+b)^2(1-2b)\lambda_r}{\lambda^2} d\lambda_r \]

\[ \lambda_r
d\lambda_r = \frac{-6b^2(1-2b)}{(3b-1)^2} db \]

\[ C_p = \frac{12}{\lambda^2} \int_{b_0}^{0.5} \left[ \frac{b(1-2b)(1+b)}{(3b-1)} \right]^2 db \]

Where: \(1/2 < b_0 < 1/3\)
Optimum Power Coefficient

\[ C_P = \left( \frac{4}{27} \right) \left\{ \frac{y_0}{(y_0 + 4)(1 - 2y_0)^2} \right\} \int_{y_0}^{y_1} \left( \frac{4}{y} - 3 - 9y - 2y^2 \right)^2 dy \]

Where: \( y_1 = 1/2 \) and \( 0 < y_0 < 1/2 \)
Optimum Power Coefficient

\[
C_p = \frac{16}{27} \left(1 - 2y\right) \left[1 + \frac{457}{1280} y + \frac{51}{640} y^2 + \frac{y^3}{160} + \frac{3}{2} y \left\{ \ln(2y) + (1 - 2y) + \frac{1}{2} (1 - 2y)^2 \right\} \left(1 - 2y\right)^3 \right]
\]
Torque Coefficient

\[ P = Q\Omega \]

\[ C_Q = \frac{C_P}{\lambda} = C_P \left( \frac{3\sqrt{3}y}{\sqrt{4 + y(1 - 2y)}} \right) \]

\[ C_Q = \frac{8}{9} \sqrt{\frac{3y}{\left(1 + \frac{y}{4}\right)^3}} \left[ 1 + \frac{457}{1280} y + \frac{51}{640} y^2 + \frac{y^3}{160} + \frac{3}{2} y \left\{ \ln(2y) + (1 - 2y) + \frac{1}{2} (1 - 2y)^2 \right\} \right] \]
Thrust Coefficient

\[ dT = 2\rho (2\pi r dr) w[U - w \cos(\phi)] \cos(\phi) \]

\[ dC_T = 8b [1 - b \cos(\phi)] \cos(\phi) \bar{r} dr = \frac{2(1 - b^2) \lambda_r d\lambda_r}{\lambda^2} \]

\[ C_T = \frac{12}{\lambda^2} \int_{b_o}^{0.5} \frac{b^2 (1 - 2b)(1 - b^2)}{(3b - 1)^2} db \]
Thrust Coefficient

\[ C_T = \frac{8}{9 \left(1 + \frac{y}{4}\right)} \left\{ 1 + \frac{55}{768} y - \frac{17}{192} y^2 - \frac{1}{64} y^3 + \frac{1}{4} y \left[ \ln(2y) + (1 - 2y) \right] \right\} \left(1 - 2y\right)^2 \]
Power Coefficient as a Function of Tip Speed Ratio

Figure 6. Power Coefficient as a Function of Tip Speed Ratio
Figure 7. Torque Coefficient as a Function of Tip Speed Ratio
Thrust Coefficient as a Function of Tip Speed Ratio

Figure 8. Thrust Coefficient as a Function of Tip Speed Ratio
Optimal Chord and Pitch

\[ c = \frac{8\pi r}{BC_l} (1 - \cos(\phi)) \quad [\text{Ref.1}] \]

\[ dL = 2\rho w[U - w\cos(\phi)](2\pi rdr) = \left( \frac{\rho B_c}{2} \right) \left[ U^2 + (\Omega r)^2 - w^2 \right] C_l dr \]

\[ 8\pi r [1 - b \cos(\phi)] = BcC_l (1 - \lambda_r^2 - b^2) \]
Optimal Chord and Pitch

\[ c = \frac{4\pi r}{BC_l} \left[ \frac{(3b - 1)}{b} \right] = \frac{16\pi rb^2}{BC_l} \left[ \frac{1}{(1 + \lambda_r^2)} \right] \]

\[ \sigma = \frac{Bc}{2\pi r} = \frac{8(1/2)^2}{C_l} = \frac{2}{C_l} \]
Optimum Chord Distribution

Figure 5. Optimum Chord, Theory with Wake Rotation vs Theory without Wake Rotation, $\lambda = 7$, $C_l = 1$, $B = 3$
Effect of Profile Drag

\[
C_l \cos(\phi) + C_d \sin(\phi) = C_l \cos(\phi) \left[ 1 + \left( \frac{C_d}{C_l} \right) \tan(\phi) \right]
\]

\[
C_l \sin(\phi) - C_d \cos(\phi) = C_l \sin(\phi) \left[ 1 - \left( \frac{C_d}{C_l} \right) \cot(\phi) \right]
\]
Effect of Profile Drag

Thrust Integral \( I_T = \frac{12}{\lambda^2} \int_{b_0}^{0.5} \frac{b^2 (1 - 2b)(1 + b)^{3/2}}{(3b - 1)^{3/2}} \, db \)

Power Integral \( I_P = \frac{12}{\lambda^2} \int_{b_0}^{0.5} \frac{b^2 (1 - 2b)^2 (1 + b)^{3/2} (1 - b)}{(3b - 1)^{5/2}} \, db \)
Effect of Profile Drag

\[ I_T = \frac{4y}{\sqrt{3} (y+4)(1-2y)^2} \left[ \left( \frac{1}{6} y^3 + \frac{4}{3} y^2 + \frac{8}{3} y - \frac{1}{3} + \frac{8}{3y} \right) \sqrt{y^2 + 4y} \right. \]

\[ \left. + 2 \ln \left( \frac{4}{(y+2) + \sqrt{y^2 + 4y}} \right) - \frac{321}{32} \right] \]
Effect of Profile Drag

\[ I_T = \frac{1}{6 \sqrt{3} (y + 4)} \left\{ \sqrt{y(y + 4)} \left( y^2 + 9y + \frac{99}{4} \right) - 48y \ln \left[ \frac{1 - \frac{(1 - 2y)}{(2 - y) + \sqrt{y(y + 4)}}}{(1 - 2y)^2} \right] + \frac{1}{3} (1 - 2y) \right\} \]

\[ + \frac{107}{3} \left\{ \frac{18y(y + 4) + \left( 45 - 32y - 8y^2 \right) \sqrt{y(y + 4)}}{36 + 113y - 38y^2 - 16y^3} + 9(7 - 2y) \sqrt{y(y + 4)} \right\} - \frac{64}{3} \left\{ \frac{\sqrt{y}}{3\sqrt{y} + \sqrt{y + 4}} \right\} \]
Effect of Profile Drag

\[ I_p = \frac{4y}{9\sqrt{3} (y+4)(1-2y)^2} \left[ \left( \frac{4}{15} y^4 + \frac{17}{15} y^3 - \frac{133}{45} y^2 - \frac{89}{9} y + 13 - \frac{104}{9y} + \frac{16}{9y^2} \right) \sqrt{y^2 + 4y} \right. \\
\left. + 8 \ln \left( \frac{(y+2)+\sqrt{y^2+4y}}{4} \right) - \frac{1023}{80} \right] \]
Effect of Profile Drag

\[ I_p = \frac{16}{27} \lambda \]

\[ I_p = \frac{27\lambda + 32\lambda^3}{54(1 + \lambda^2)} \]

\[ C_{TT} = C_T [Eq.(41)] + \left( \frac{C_d}{C_l} \right) I_T \]

\[ C_{PN} = C_P [Eq.(35)] - \left( \frac{C_d}{C_l} \right) I_P \]
Thrust Coefficient as a function of Tip Speed Ratio Including the Effect of Profile Drag

Figure 10. Thrust Coefficient as a Function of Tip Speed Ratio Including the Effect of Profile Drag
Power Coefficient as a function of Tip Speed Ratio Including the Effect of Profile Drag

Figure 9. Power Coefficient as a Function of Tip Speed Ratio Including the Effect of Profile Drag
Summary & Conclusions

• An alternate derivation is provided for the parameters of an optimum, ideal wind turbine. Unlike previous derivations, only a single momentum theory is used (in the direction of the local lift) so that there are no separate accounts of axial and angular momentum.

• The results, also unlike previous results, are found in closed form for all variables—and the singularities of previous numerical solutions are eliminated explicitly. Although the final parameters for the optimum turbine are no different from those of conventional approaches, the closed-form nature of the results yields insight into the properties of the optimum turbine.
Summary & Conclusions

• Finally, because of the single momentum balance, it is quite straightforward also to write a closed-form expression for the optimum blade chord distribution. The true optimum does not become singular at the blade root, but rather approaches a combination of solidity and pitch angle that avoids blade-to-blade interference.