CIRCUITS LABORATORY

EXPERIMENT 5

Circuits Containing Inductance

5.1 Introduction

Inductance is one of the three basic, passive, circuit element properties. It is inherent in all electrical circuits. As a single, lumped element, inductors find many uses. These include as buffers on large transmission lines to reduce energy surges, on a smaller scale to serve a similar function in electronic circuits, as elements in frequency selective filters in telecommunication circuits, as momentary energy storage devices in power supplies that convert power from one voltage level to another, and as devices for exerting mechanical force in electromagnets and similar electromechanical devices.

Inductors are unique in that they can be magnetically coupled such that a time-varying current in one will cause a voltage to be generated in a second inductor in close proximity. This ‘mutual inductance’ is the basis for the electrical transformer that is ubiquitous in the electric power industry. Transformers, with their impedance transforming property are also useful in electronic circuits over almost the entire frequency spectrum. We will not cover all these uses in this experiment but will mainly concentrate on the resonant circuit with inductor and capacitor, and on the measurement of mutual inductance between two air-core inductors.

5.2 Objectives

In this experiment the student should learn:

(1) How to measure the output impedance of a signal source,

(2) The circuit representation of an inductor,

(3) The definition of ‘quality factor’ or Q of a reactive element or circuit,

(4) The characteristics of a series resonant circuit,

(5) The characteristics of a parallel resonant circuit,

(6) Measurement of mutual inductance as a function of separation distance, and

(7) A data reduction method for comparing theoretical expectations with experimental results.

5.3 Function Generator Properties

5.3.1 Output Impedance

The Hewlett-Packard 33120A function generator is our signal source in this exercise. It can supply a sine, square, triangular, or unsymmetrical square wave (pulse) over a frequency range of about 0.01 Hz to 5 MHz. Its peak-to-peak output voltage with open circuit load is adjustable from less than 0.2 V to 60 V, and once set, the output magnitude on open circuit is essentially constant as the frequency is varied.

The 33120A is not a perfect signal source. We may represent it in circuit form as shown in figure 5.1.

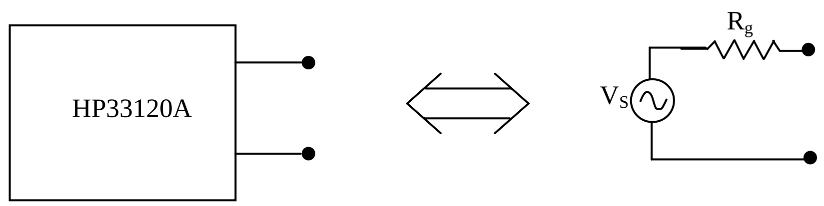


Figure 5.1: Function generator representation.

One of the things we will do in the experimental part of this exercise is to determine the value of R_g for the HP33120A. Several methods are available to do this. Perhaps the simplest is to simply set the generator voltage to a reasonable value, V_S, on open circuit. A resistor, R_L, is then placed across the generator terminals and, as can be seen

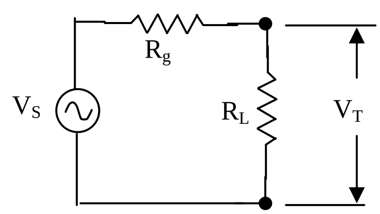


Figure 5.2: Function generator with load resistor.

from figure 5.2, the terminal voltage will decrease to a value, V_T, where

$$V_T = \frac{R_L}{R_g + R_L} V_S \tag{5.1}$$

or

$$R_g = R_L \left(\frac{V_S}{V_T} - 1\right) \quad . \tag{5.2}$$

R_g is termed the 'internal impedance' of the generator. Note that, if R_g were an impedance, Z_g, with a resistive and a reactive part, the measurement method above would not yield the correct value for even the $|Z_g|$, let alone the resistive and reactive parts. A more complex experimental procedure would need to be used in this case, possibly involving termination with several different values of resistance and reactance.

If the exercise is properly done, the value of R_g obtained for the HP33120A should be about 50Ω. In equipment designed for use at high frequencies, best performance is obtained if the output and input impedance of interconnected apparatus is the same value. Partly due to the physical characteristics of cables and partly due to convention, this impedance level has been standardized at 50Ω. There are exceptions, however. For many years, telephone and some audio apparatus, whose proper operation also requires impedance matching, has standardized on the value of 600Ω.

5.4 Inductors

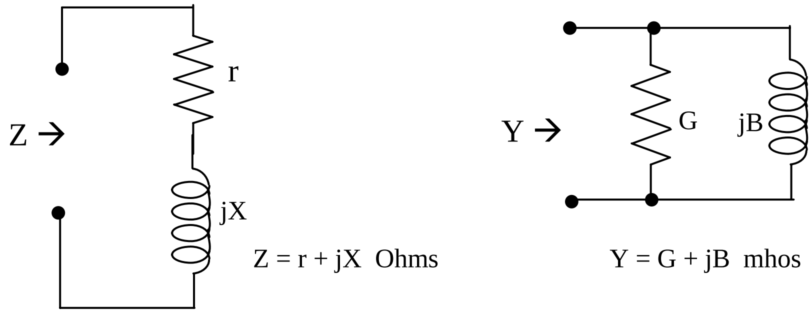
All electrical circuits possess inductance to a greater or lessor degree. Commonly, a circuit element that is primarily inductive can be formed by a coil of wire. The inductance can be enhanced if the coil links material with a high magnetic permeability such as soft iron, laminated steel, powdered iron, or ferrite. In this exercise we will use an air core coil, i.e., one that has no magnetic material in its interior.

A two-terminal element, such as an inductor coil at a particular frequency, has an impedance given by

$$Z = \mathrm{r} + \mathrm{j}X \tag{5.3}$$

Both r and X will generally be functions of frequency. If X has a positive value, we say the element is 'inductive' at that frequency. It should be noted that the expression $Z = \mathrm{r} + \mathrm{j}X$ implies a series representation of the element with a resistor and inductor (if $X > 0$) in series. A parallel representation is equally valid at a single frequency.

Figure 5.3 shows the relationship between the two representations.



$$G = \frac{r}{r^2 + X^2} \qquad B = -\frac{X}{r^2 + X^2} \tag{5.4}$$

Figure 5.3: Series-parallel equivalence

Considering that a wire-wound inductor has an inherent resistance due to the resistance of the wire itself, the series representation of the inductor with its unavoidable resistance seems the most natural and gives elements whose frequency variation is simpler than if the parallel representation were used. Typically, X for an inductor will vary with radian frequency, ω, as

$$X = \omega L. \tag{5.5}$$

This is valid up to some upper frequency limit, where the inter-turn capacitance of the coils in the inductor cause $dX/d\omega$ to become larger than L. At some frequency in this range, the coil will be ‘self-resonant’ and its reactance will be capacitive rather than inductive, at frequencies larger than the self-resonant frequency.

Coil resistance also varies with frequency. This is because at higher frequencies current exists primarily on the surface of conductors rather than the interior and so the resistance of a conductor increases as frequency increases, although not linearly as is the case with inductive reactance.

A quantity that is commonly used to characterize an inductor is the 'quality factor', abbreviated as 'Q'. If the coil is represented as a series resistor, r, and reactor, *X*, then

$$Q = \frac{X}{r} \quad . \tag{5.6}$$

Wire-wound inductors with a substantial number of turns will have *Q* values in the range of 10 to 50. Since both *X* and r vary with frequency, so also will the *Q* value and there will be a frequency at which the coil *Q* has a maximum value. It should be mentioned that *Q* has a broader definition than the one given above. More generally, for an oscillating system

$$Q = \frac{2\pi(\text{Energy stored})}{\text{Energy dissipated / cycle}} \quad . \tag{5.7}$$

5.5 Series Resonance

Figure 5.4 shows a circuit with an inductor, *L*, and a capacitor, *C*, connected in series. Also in series is a signal source, V_S, with its associated output resistance, R_g, and, possibly, an external added resistor, R_e.

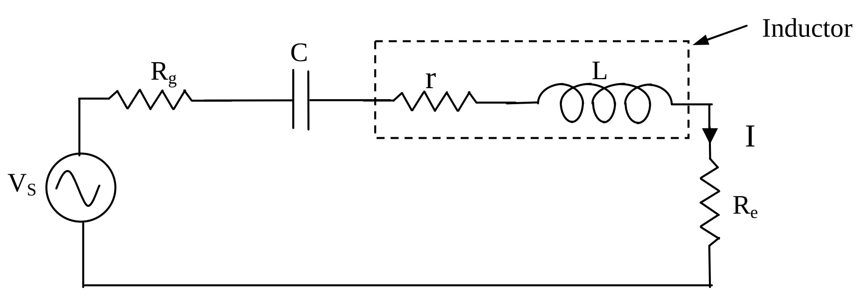


Figure 5.4: Series resonant circuit

The capacitor, C, is assumed to be lossless. This is not strictly true, and, indeed, just as for an inductor a Q value could be ascribed to a capacitor to account for its loss. However, a capacitor of reasonable quality can easily have a Q value exceeding several hundred and so for simplicity we will neglect any capacitor loss here.

It is straightforward to compute the value of the current, I, in this circuit. We have

$$I = \frac{V_S}{(R_g + R_e + r)} \quad \frac{1}{1 + jQ_C\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \tag{5.8}$$

where

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad Q_C = \frac{1}{(R_g + R_e + r)}\sqrt{\frac{L}{C}} \quad . \tag{5.9}$$

Note that ω_o is the resonant natural frequency and Q_C is the "quality factor" of the circuit A graph of I versus frequency gives a "resonance curve" with its characteristic bell shape showing the peak value and the bandwidth (BW).

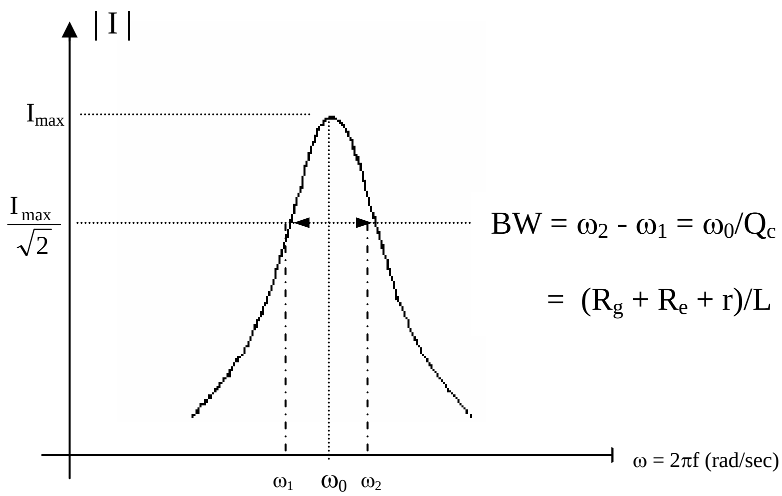


Figure 5.5: Resonance curve for series resonant circuit

From Equation 5.8 it is clear that $$I_{\max} = \frac{V_g}{R_g + R_e + r} \tag{5.10}$$

As defined here, Q_C is the circuit Q and is determined by the total series resistance in the circuit. Note that the selectivity or relative narrowness of the resonance curve is governed by the value of Q_C. Higher values of Q_C imply a narrower curve or greater selectivity in the frequency range to which the circuit is responsive. In the event that the total series resistance were at its minimum value, namely just the resistance, r, inherent in the inductor, then the Q_C of the resonant circuit would be equal to the inductor Q. Added circuit resistance causes the circuit Q_C to be less than the inductor Q.

5.6 Parallel Resonance

Connecting an inductor and a capacitor in parallel gives a second type of resonant circuit. The major features of parallel resonance are best illustrated by the idealized circuit of figure 5.6.

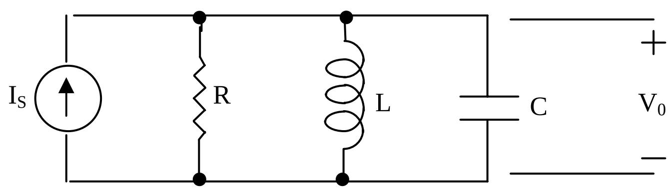


Figure 5.6: Idealized parallel resonant circuit

Here, we may verify that the voltage across the circuit, V_0, is given by:

$$V_0 = \frac{I_S R}{1 + jQ_C\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \tag{5.11}$$

where

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad Q_C = \frac{R}{\sqrt{\frac{L}{C}}} \tag{5.12}$$

Plotting V_0 vs. frequency in figure 5.7 gives a resonance curve similar to that for the series case.

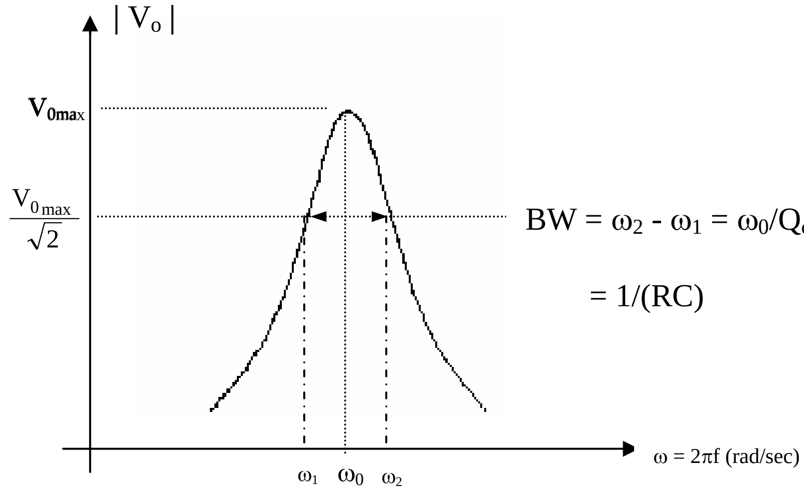


Figure 5.7: Resonance curve for parallel resonant circuit

From Equation 5.11, it is apparent that the voltage across the circuit, V_0, is maximum at the frequency, ω_0, and that the maximum value of V_0 is

$$V_{0\max} = R\, I_g. \tag{5.13}$$

One difference between parallel and the series resonant circuits is the Q_C value, which determines the circuit bandwidth or selectivity. For the parallel circuit, Q_C is

$$Q_{C(\text{parallel})} = \frac{\text{Parallel resistance}}{\text{Resonant reactance}} \tag{5.14}$$

while Q_C for the series circuit is

$$Q_{C(\text{series})} = \frac{\text{Resonant reactance}}{\text{Series resistance}} \tag{5.15}$$

The general definition for Q given by Equation 5.7 encompasses both of these cases.

Of course, in a parallel resonant circuit with an actual inductor, the inductor has a resistance that must be taken into account. This can be accomplished by using the series-parallel transformation given previously and shown below.

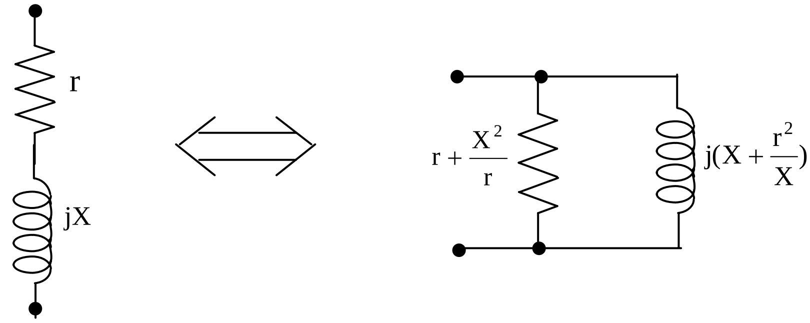


Figure 5.8: Series-parallel transformation, all elements are resistors or reactors.

We will assume that, over the frequency range where the inductor is being used, *X*/r is relatively large, say > 10, so that there is little error is in simplifying the transformation to

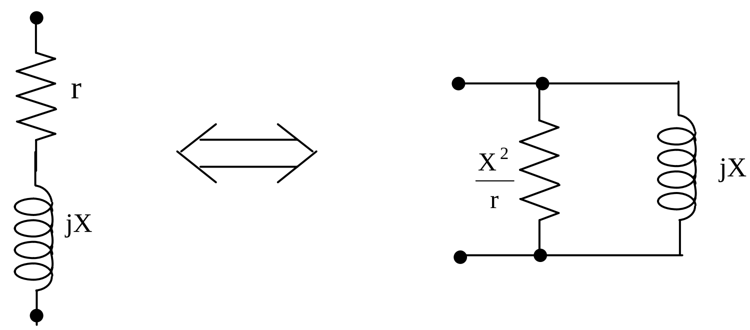


Figure 5.9: Simplified approximate series-parallel transformation.

A numerical example can be used to illustrate. Consider a 10mH inductor with a quality factor *Q* of 30 at ω_0. It is connected in parallel with a 0.025μF capacitor, which combination is in series with a 15kΩ resistor. When driven by a voltage source of negligible output impedance, what will be the relative variation of voltage with frequency across the *L-C* circuit near resonant frequency? Figure 5.10 shows the original circuit and the equivalent one using the simplified transformation of Figure 5.9. Note that for ω_o = 63,245 rad/sec, the element values are X = 632.5 Ω, r = 21.08 Ω, and X^2/r = 19.0 kΩ.

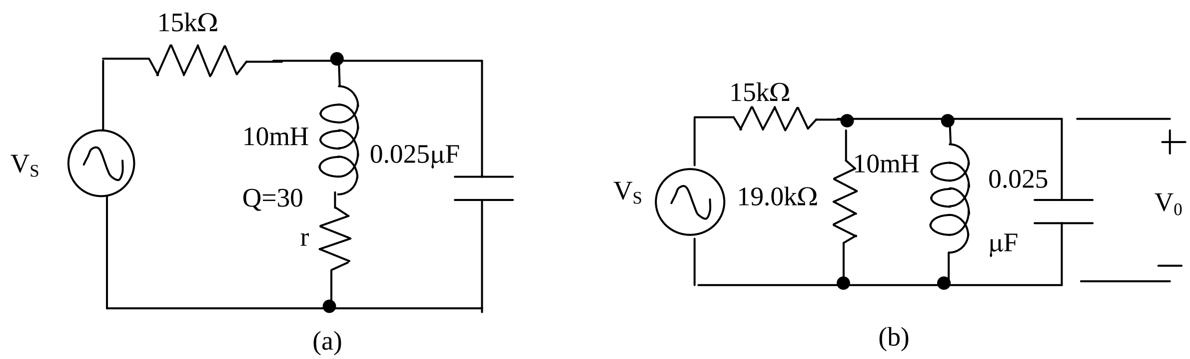


Figure 5.10: Original Circuit (a) and Simplified Equivalent Circuit (b).

Using Norton's theorem allows circuit (b) to be drawn in the idealized form of Figure 5.6 with R = 15 kΩ//19 kΩ = 8.38 kΩ. The plot of relative output voltage vs. frequency is shown in Figure 5.11.

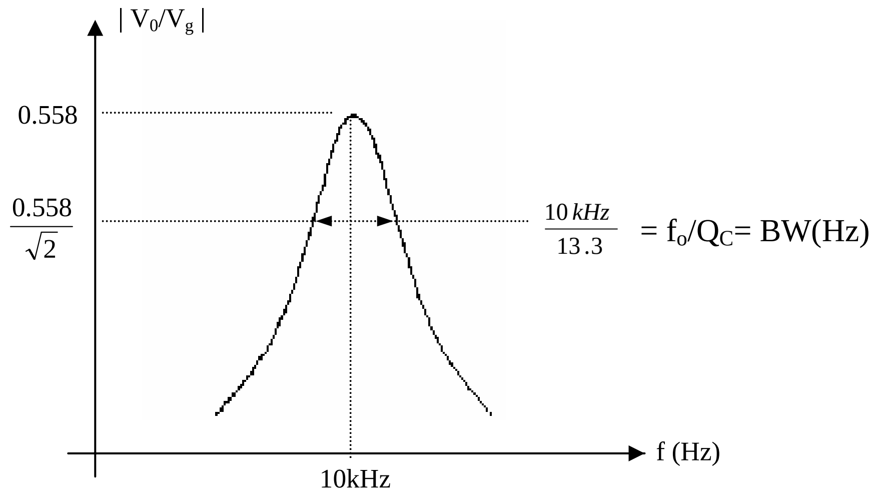


Figure 5.11: Numerical example result.

5.7 Inductive Coupling or Mutual Inductance

5.7.1 Mutual Inductance

In a region of space where currents exist, if there is no magnetic material present, the magnetic flux is linearly proportional to the currents in the region. Consider a number of current meshes labeled (1, 2, ...,j, ...J) carrying mesh currents $i_j(t)$ and linking fluxes $\Psi_j(t)$. Then

$$\Psi_k(t) = L_{k1}i_1(t) + L_{k2}i_2(t) + ... + L_{kj}i_j(t) + ... + L_{kJ}i_J(t) = \sum_{j=1}^{J} L_{kj}i_j(t) \qquad (5.16)$$

In Eq. 5.16 the coefficients L_{kj} ($j \neq k$) are termed the 'mutual inductances' of the k*th* mesh while L_{kk} is called the 'self-inductance'. Conventionally, the mutual inductances are denoted by the letter M while the letter L is reserved for self-inductance.

There will be induced in mesh k a voltage of the form

$$v_k = \frac{\partial \Psi}{\partial t} = \sum_{j=1}^{J} L_{kj} \frac{di_j(t)}{dt} \qquad (5.17)$$

It is clear that the magnitude of this voltage depends on several factors, which include geometric orientation, coil spacing, and current magnitudes.

5.7.2 Mutual Inductance Between Two Small Circular Loops Widely Separated

The mutual inductance between two circular loops of average radius a, aligned along the same axis and perpendicular to that axis, is rather simply approximated when the loops are sufficiently far apart. If the separation distance of the centers of the two coils is z, then the value of this inductance is approximately

$$M(z) \cong M_0 \frac{a^3}{(a^2 + z^2)^{3/2}} \qquad (5.18)$$

where

$$M_0 \approx \frac{\mu_0 \pi a n^2}{2} \qquad (5.19)$$

and $\mu_0 = 4\pi(10)^{-7}$ Henries/meter, further

n is the number of coil turns, and

a is the radius of the coil .

5.7.3 A Two-coil Circuit with Mutual Inductance

Consider the coupled circuit shown in Fig. 5.12. Kirchhoff's current law for these two

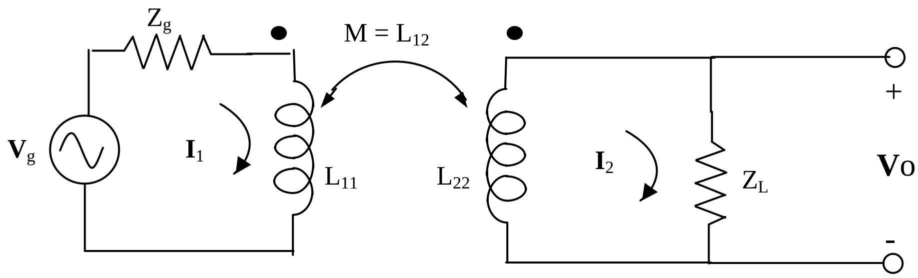


Figure 5.12: Circuit with two magnetically coupled coils.

meshes can be written down at once :

$$\mathbf{V}_g = Z_g \mathbf{I}_1 + j\omega L_{11}\mathbf{I}_1 - j\omega M\mathbf{I}_2 \quad (5.20)$$

$$0 = -j\omega M\mathbf{I}_1 + Z_L \mathbf{I}_2 + j\omega L_{22}\mathbf{I}_2. \quad (5.21)$$

The output voltage $\mathbf{V}_o$ across Z_L is just I_2Z_L. So:

$$\mathbf{V}_o = \mathbf{I}_2 Z_L = V_g Z_L \frac{j\omega M}{(Z_g + j\omega L_{11})(Z_L + j\omega L_{22}) + \omega^2 M^2} \quad (5.22)$$

If $\omega L_{22} << |Z_L|$ and $Z_gZ_L >> \omega^2M^2$ or if the secondary circuit is open ($\mathbf{I}_2 = 0$), then

$$|\mathbf{V}_O| \cong \omega M \frac{V_g}{|Z_g + j\omega L_{11}|} = \omega M|\mathbf{I}_1| \quad . \quad (5.23)$$

Eq. (5.23) is the so-called 'weak-coupling' limit. It expresses the commonly observed physical reality that a large (relatively speaking) current in a 'primary winding' can induce a significant voltage in a 'secondary' winding without being affected by the resulting current in the secondary winding.

There is also a 'strong-coupling' limit of Eq. (5.22). If $\omega L_{11} >> |Z_g|$, $L_{11}L_{22} = M^2$, and $|Z_L| >> \omega L_{22}$, then

$$|\mathbf{V}_o| = |\mathbf{I}_2\mathbf{Z}_\mathbf{L}| \cong \mathbf{V}_\mathbf{g}\sqrt{\frac{\mathbf{L}_{22}}{\mathbf{L}_{11}}} \quad (5.24)$$

Eq. (5.24) applies to that most valuable linchpin of modern civilization, the power transformer. Unfortunately, we will not have time to investigate it in this course.

5.7.4 Linearizing Transformations

This is a complicated topic in all of its advanced details, but the philosophy that underlies it is simple enough. Consider two variables, x and y, whose values are associated, and let N data pairs (x_n, y_n) be experimentally collected. Assume that, over an interval (X_a, X_b), there is a theoretical monotonic relation between x and y such that

$$y = f(x; \alpha, \beta) \qquad X_a \leq x \leq X_b \;, \quad \alpha, \beta \text{ constants} \;. \tag{5.25}$$

The problem is to use the observed data to determine the most likely values of the constants, α and β.

We define two transformations $u = u(x,y)$ and $v = v(x,y)$ such that Eq. (5.25) is transformed into the familiar linear slope-intercept form given as

$$v = m\,u + b, \tag{5.26}$$

where

$$m = m(\alpha, \beta) \tag{5.27}$$

$$b = b(\alpha, \beta). \tag{5.28}$$

In this way the data set (x_n,y_n) $(n = 1, \ldots N)$ is transformed into the set (u_n,v_n), the members of which should plot along a straight line. The great advantage of the transformation to the variables u and v is that a simple linear regression (or a quick sketch with a straight edge) passes a ‘best fit’ straight line through the transformed points and yields estimates of m, the slope, and b, the intercept on the v axis, from which the values of α and β can be deduced.

As an example, consider the case where x and y are related by the equation:

$$y = \alpha x^{\beta} \ . \tag{5.29}$$

Taking the logarithm of both sides yields

$$\ln(y) = \beta \ln(x) + \ln(\alpha) \ .$$

Here, $v = \ln(y)$, $u = \ln(x)$, $m = \beta$, and $b = \ln(\alpha)$.

As another example, suppose the theoretical relation between x and y is

$$y = \frac{\alpha x}{x + \beta} \ . \tag{5.30}$$

This equation can be linearized in several different ways. They are

a) $$\left\{\frac{1}{y}\right\} = \frac{1}{\alpha} + \frac{\beta}{\alpha}\left\{\frac{1}{x}\right\} \tag{5.31a}$$

b) $$\left\{\frac{x}{y}\right\} = \frac{\beta}{\alpha} + \frac{1}{\alpha}\{x\} \tag{5.31b}$$

c) $$\{y\} = \alpha - \beta\left\{\frac{y}{x}\right\} \tag{5.32b}$$

It is apparent from this latter example that a number of data linearizing transformations may exist for a given theoretical relation.

5.8 Experimental Procedure

5.8.1 Equipment List

1 Test station with standard equipment

1 Clamp stand with swivel holder

2 J. W. Miller 990 inductors (nominal R <318Ω, L = 47 mH)

1 Plastic rod (18"long by 3/8"diameter) with one of the inductors affixed.

1 Meter stick.

5.8.2 Function Generator Output Impedance

Use a DMM to measure the actual resistance of the 47 Ω, 1 W, resistor and record this value. Using a 10x probe, set the open circuit output voltage (V_{go}) of the HP33120A function generator to a 8 V_{p-p} sine wave of 100 Hz. Next, connect the 47Ω resistor across the output terminals of the function generator. Measure the voltage across the resistor using a 10x probe and record its value. Repeat for 1 kHz, 10 kHz, and 100 kHz.

5.8.3 Series Resonance

3.1 Measure with the DMM and record the DC resistance of the 47mH inductor. Compute the value of capacitor needed to resonate with the inductor at f_a. Construct a series resonant circuit consisting of the function generator, a 100Ω, 1 W resistor, one of the 47mH inductors, and a fixed lumped capacitor. Use the nearest single standard size fixed capacitor available in the laboratory for this circuit.

3.2 With its output displayed on Channel 1, set the function generator open circuit voltage to 8 V peak-peak at a frequency of f_a. Display the voltage across the 100 Ω resistor on Channel 2 and use the X-Y display function on the scope to find and record the actual resonant frequency f_0 by adjusting the function generator frequency slightly above or below f_a. Now calculate the theoretical circuit bandwidth (BW) in Hz and take data at evenly spaced frequencies from one BW below f_0 to one BW above f_0 to clearly delineate the resonance curve. Record in a table the Channel 1 and Channel 2 voltages at each of these frequencies. Also, locate and record the voltages at the resonant frequency f_0 and at the two half-power point frequencies (f_1 and f_2).

3.3 Repeat Step 3.2 with a 1000Ω series resistor.

5.8.4 Parallel Resonance

4.1 Using the capacitor that you used in 5.8.3 above, a 100 kΩ resistor, the 47mH inductor, and the function generator, construct a parallel resonant circuit similar to that in Figure 5.10(a) where the inductor and capacitor are in parallel and this combination is in series with the 100 kΩ resistor and the function generator. Again set the function generator open circuit voltage to 8V peak-peak and frequency f_a. Using the 10x probe to display the capacitor voltage on Channel 2, find and record f_0 using the X-Y display function. Calculate the theoretical BW in Hz and record in a table the Channel 1 and 2 voltages over the frequency range from one BW below f_0 to one BW above f_0 to clearly delineate the resonance curve. Also, locate and record data at f_0 and at the two half-power point frequencies (f_1 and f_2).

4.2 Repeat 4.1 above, except now use a 20 kΩ series resistor.

5.8.5 Mutual Inductance

5.1 Use a clamp-stand to hold the rod with the permanently affixed inductor and connect it to the HP33120A function generator through a DMM ammeter. Connect the

DMM voltmeter across the second movable inductor.

5.2 Set the function generator to its maximum sine wave output at a frequency of f_b as given by your instructor. Using the two DMMs to measure the primary current I_1 and secondary voltage V_2, take sufficient data to determine M(z), where ***z is the separation distance of the centers of the two coils***. Note that z is approximately 1.3 cm when the faces of the two coils touch.

5.9 Report

5.9.1 Output Impedance

1.1 From your experimental data, compute the output resistance of the function generator at all four frequencies. Considering the fact that reactance varies with frequency, what do these calculated resistances tell you about the nature of the impedance of the function generator? Are your results in agreement with the manufacturer's value of 50 Ω? What are the % difference between your results and the specified 50 Ω?

1.2 Why will this method give erroneous results if the output impedance is not purely resistive? Suggest another method that might give improved results for this case.

5.9.2 Series Resonance

2.1 Plot your current data versus frequency in Hz for the two values of external series resistor that you used. Use a **linear frequency scale** chosen to give a resonance curve over twice the calculated bandwidth. Compare the measured bandwidth with the calculated bandwidth for both cases. Are your results reasonable?

2.2 Determine Q_C for the circuit from the experimental data for the above two cases.

2.3 From the measured bandwidth obtained in 2.1 above, calculate values for the series resistance of the inductor at resonant frequency for both cases assuming the inductance is 47 mH. Compare these resistances with the DC value measured with the ohmmeter. Which is larger?

2.4 Calculate the Q of the inductor at resonant frequency from each of your two sets of data. Show your calculations.

5.9.3 Parallel Resonance

3.1 Plot your voltage data versus frequency for the two values of external series resistor that you used. Use a **linear frequency scale**. Compare the measured bandwidth with the calculated bandwidth for both cases. Are your results reasonable?

3.2 Determine Q_C for the circuit from the experimental data for the above two cases.

3.3 From the results of obtained in 3.1, calculate values for the series resistance of the inductor at resonant frequency for both cases assuming the inductance is actually 47 mH. Are the results in agreement with those obtained for series resonance?

3.4 Calculate the Q of the inductor at resonant frequency from each of your two sets of data. Show your calculations. Do your results agree with the Q value as calculated from the series resonance data?

5.9.4 Mutual Inductance

4.1 Use your data to derive values of the mutual inductance, M(z). Present the results in tabular form, i.e., M(z) versus z.

4.2 Design a suitable linearizing transformation to demonstrate that Eq. 5.18 is at least qualitatively correct. From your linear plot, deduce a value for the coil radius, *a*.

5.9.5 Design Problem

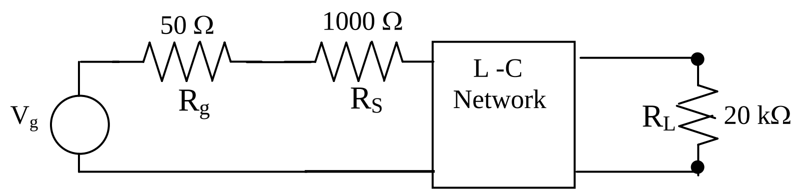


Figure 5.13: Design Problem Circuit.

In Figure 5.13, the generator is sinusoidal at a frequency f specified by the instructor. Design a L-C network using a single inductor L and single capacitor C to obtain maximum power transfer to the resistive load R_L. Assume the inductor is lossless for design purposes. Hint: Assume a series inductor followed by a shunt capacitor, transform the resulting parallel $R_L C$ circuit into a series circuit, and apply the criteria for series resonance to obtain maximum power transfer.

Document your design by providing the following:

5.1 A circuit diagram that includes the generator, resistors, L-C network, and the load,

5.2 The values selected for the inductor L and the capacitor C,

5.3 The power delivered to the resistive load assuming V_g = 10 Vrms,

5.4 The power delivered to the resistive load if the inductor is not lossless, but instead has a "Q" of 20 at the specified frequency f.

5.10 References

1. Nilsson, J. W., **Electric Circuits**, (6th ed.), Prentice Hall, Upper Saddle River, New Jersey, 2001.

2. Terman, F. E., Electronic and Radio Engineering, McGraw-Hill, New York, NY, 1955.