CIRCUITS LABORATORY

EXPERIMENT 4

Characteristics of Periodic Waveforms

4.1 Introduction

Waveforms encountered in electronic circuits are often characterized by a variety of parameters that succinctly summarize important features. Examples include the peak value, average value, and effective value of the waveform. Such parameters can be used for periodic waveforms, pulse waveforms, and even random or noise waveforms.

In this experiment, you will measure some important parameters of periodic waveforms in simple electronic circuits. The measurements will be compared to the parameter values predicted by analysis of the waveforms. As a result of exercise, you will gain a thorough understanding of how to measure and analyze periodic wave forms with the equipment available in our lab.

4.2 Objectives

At the end of this experiment, the student will be able to:

(1) determine the Fourier series and RMS value for an arbitrary periodic waveform,

(2) use the oscilloscope to measure periodic waveforms in both the DC coupled and AC coupled modes, understanding the difference between these two modes,

(3) measure the RMS voltage of a periodic waveform using only the DMM,

(4) develop a polynomial approximation to fit a particular set of data,

(5) use Parseval's equation to predict the RMS voltage of a periodic waveform,

(6) perform simple power calculations, and

(7) design a simple rectifier through the use of a diode.

4.3 Theory

4.3.1 Power and Energy for Arbitrary Waveforms

Shown in Figure 4.1 is a two-terminal electrical element with voltage, v(t), and current, $i(t)$, having the assigned polarities. This could be a resistor, capacitor, inductor, or even a combination of these with other elements to form an electrical circuit with two access terminals. The instantaneous power, $p(t)$, absorbed by the element is by definition the product of the voltage and current, $p(t) = v(t)i(t)$.

For an ideal resistor of resistance R, $v(t) = Ri(t)$, so the instantaneous power for a resistor may be expressed in terms of the resistance and either the current or

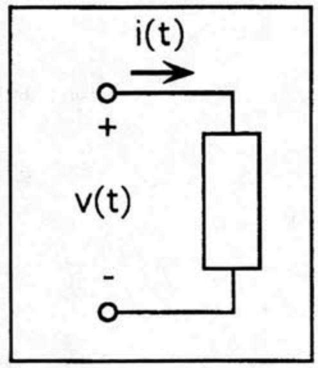

Figure 4.1: A two-terminal electrical element

voltage alone according to

$$p(t) = Ri^2(t) = v^2(t)/R. \tag{4.1}$$

Since $R > 0$, it follows for a resistor that the instantaneous power is always positive, which means that a resistor can only absorb power. The power absorbed by a resistor is converted into heat, which is dispersed into the surrounding environment. The temperature of the resistor will rise until an equilibrium point is reached where the power being absorbed equals the power being dispersed. If the material of which the resistor is made and the geometry of the resistor are such that too high a temperature is required for equilibrium, the resistor "burns out" and the continuity of the circuit branch containing it is broken so the resistor is just a useless blown fuse! For this reason, commercial resistors are rated according to the maximum power they can absorb; a resistor with a one watt rating is larger than one having a one-half watt rating because, for the same material, a greater surface area is needed to disperse the greater absorbed power into the environment.

We can conduct a similar analysis of the instantaneous power for other circuit elements using the corresponding i-v characteristics. The i-v characteristic for an ideal capacitor, *C*, is

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \qquad (4.2)$$

or, alternatively,

$$i(t) = C\frac{dv(t)}{dt}\,, \qquad (4.3)$$

so the instantaneous power absorbed by a capacitor may be expressed in terms of the capacitance and either the current or voltage alone according to

$$p(t) = \frac{1}{C}i(t)\int_{-\infty}^{t} i(\tau)d\tau = Cv(t)\frac{dv(t)}{dt} = \frac{1}{2}C\frac{dv^2(t)}{dt} \qquad (4.4)$$

The derivatives and integrals appearing in these expressions make the evaluation of the instantaneous power absorbed by a capacitor somewhat more complicated than that for a resistor where only simple algebraic evaluations are required. There is a more important implication, however. The instantaneous power absorbed by a capacitor can be positive or negative. As an example, suppose that the voltage is given by $v(t) = V_m \sin(2\pi ft)$. Let $A = V_m$.

Then,

$$p(t) = 2\pi fCA^2 \sin(2\pi ft)\cos(2\pi ft) = 2\pi fCA^2 \sin(4\pi ft)\,. \qquad (4.5)$$

This indicates that the instantaneous power for a capacitor can be positive some of the time and negative at other times. The interpretation is that the capacitor is absorbing power when $p(t) > 0$ and is delivering power when $p(t) < 0$. An inductor can also absorb and deliver power, as you can discover by similar arguments using its i-v characteristic.

The energy, $w(t_1,t_2)$, absorbed in a time interval (t_1,t_2) by the element in Figure 4.1 is by definition the integral of the instantaneous power over the interval (t_1,t_2), i.e.,

$$w(t_1,t_2) = \int_{t_1}^{t_2} p(t)dt \ . \tag{4.6}$$

This energy is always positive for a resistor, but it can be negative for a capacitor and other two-terminal elements depending on the sign of $p(t)$ and the interval of interest.

The energy in a time interval divided by the duration of this interval has the units of power and is termed the average power, P_{AVE}, over the interval. Thus,

$$P_{AVE} = \frac{w(t_1,t_2)}{t_2 - t_1} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)dt \ . \tag{4.7}$$

4.3.2 Power and Energy for Periodic Waveforms

Now we will examine the situation when the voltage and current waveforms are both periodic functions of time. To do this, let us reconsider the element of Figure 4.1. If this element is linear, so that the principle of superposition holds, and it is time-invariant, so that its properties do not change with time, and if the voltage waveform $v(t)$ is a periodic, sinusoidal function of time, then the current waveform $i(t)$ is also a periodic, sinusoidal function of time with the same frequency as the voltage. That is, if $v(t)$ is of the form $A \sin(2\pi ft)$, then $i(t)$ will be of the form $B \sin(2\pi ft + \theta)$. Thus, the frequency f is the same, but the amplitude and phase may be different depending on the properties of the element. For example, $B = AR$ and $\theta = 0$ for a resistor, and $B = 2\pi fCA$ and $\theta = \pi/2$ for a capacitor, as may be seen from the i-v characteristics for these elements.

More can be said when the element is both linear and time-invariant and the voltage waveform is a periodic function of time. To do so, we will introduce the concept of a Fourier series expansion, which can be applied to any periodic waveform. Recall that a waveform $x(t)$ is periodic if $x(t \pm nT) = x(t)$ for all t. The smallest such T that satisfies this relationship is called the fundamental period. Through the investigation of heat-flow problems, French mathematician Jean Baptiste Fourier (1768 - 1830) discovered that a periodic function can be represented by an infinite sum of harmonically related sinusoids. That is, the period of any sinusoid in the infinite series is an integral multiple, or harmonic, of the fundamental period T of the periodic waveform. So, given that $v(t)$ is periodic with fundamental period T, then $v(t)$ can be expressed as a Fourier series according to

$$v(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi f_n t) + \sum_{n=1}^{\infty} b_n \sin(2\pi f_n t) \ , \qquad (4.8)$$

where $f_0 = 1/T$ is the fundamental frequency, and $f_n = nf_o$, the integral multiple of f_0, are known as the harmonic frequencies (or simply harmonics) of $v(t)$. The coefficients a_n and b_n are given by

$$a_0 = \frac{1}{T}\int_0^T v(t)dt \ , \qquad (4.9)$$

and, for $n \geq 1$,

$$a_n = \frac{2}{T}\int_0^T v(t)\cos(2\pi f_n t)dt \ , \qquad (4.10)$$

$$b_n = \frac{2}{T}\int_0^T v(t)\sin(2\pi f_n t)dt \ . \qquad (4.11)$$

In examining the equation for a_0, note that this value is simply the average value of the waveform $v(t)$.

We can make a number of general comments about the use of the Fourier series expansion. The first concerns the question of what conditions, if any, must be satisfied by the periodic waveform $v(t)$ in order for it to have a convergent Fourier series. The precise mathematical answer to this question is somewhat complicated, but it is sufficient to say that all periodic waveforms of interest can be expressed in terms of a Fourier series. That is, any periodic waveform that is generated by a physically realizable source will have a convergent Fourier series.

The second observation that one can make is that once $v(t)$ is known and the Fourier coefficients (a_n, b_n) have been calculated, the periodic waveform has effectively been decomposed into a dc source (a_0) plus a sum of sinusoidal sources (a_n, b_n). This fact has an important implication and is the reason why the Fourier series is an important tool in circuit analysis. Since the waveform $v(t)$ is driving a linear circuit, one can use the principle of superposition to find the steady-state response. That is, one calculate the response to each of the individual sinusoidal source in the Fourier series expansion, and then add all of these responses to get the total response. In general, the best way to calculate the steady-state response to an individual sinusoidal source is to use phasor analysis, although other techniques could be used.

At this point, one may be concerned about the feasibility of calculating the response of a circuit to a periodic waveform using this method. After all, in general, a Fourier series will have an infinite number of sinusoidal terms, which in theory could lead to an infinite summation that could not be solved analytically. However, in practice this is not as severe a limitation as it might seem. The coefficients of a Fourier series, a_n and b_n, usually decrease rapidly as n increases, so that in practice one need only consider the first few terms to obtain reasonable accuracy. Furthermore, through the use of computers, one can calculate as many terms as are needed to achieve the desired accuracy.

As a final observation, note that the method of finding the steady-state response to a periodic signal is straightforward and involves no new techniques of circuit analysis. However, as was briefly noted in the above paragraph, this method of circuit analysis does have some drawbacks. For one thing, it produces the Fourier series representation of the steady-state response, with the actual shape of the response being unknown. And again, as noted above, the actual response can only be estimated by adding a sufficient number of terms. Yet, despite its drawbacks, this method can produce useful quantitative results, and it introduces a conceptual way of thinking about the problem that will be important in analyzing more complex systems.

Now let us return to the circuit element in Figure 4.1 and again assume that this element is linear and time invariant and that the voltage waveform is a periodic function of time. Since a periodic waveform can be expressed as a linear combination of harmonically related sinusoids by using a Fourier series expansion and since the element is linear and time invariant, it follows that the current will also be a linear combination of harmonically related sinusoids, each having the same frequency as the corresponding harmonic of the voltage waveform, but having a different amplitude and phase. This implies that the current waveform will also be a periodic function of time with exactly the same period as the voltage waveform. Moreover, since the instantaneous power is the product of two waveforms with the same period, it too will be a periodic waveform with the same period, although its fundamental period may differ from the fundamental period of the current and voltage waveforms.[1]

Because of these reasons, it is both natural and common with periodic waveforms to select an interval (t_1, t_2) of duration $T = t_2 - t_1$ equal to a period of the instantaneous power in the definition of the average power dissipated by a linear, time-invariant element. It should be evident that the same average power is obtained for any period selected in forming this average, including the fundamental period or any integer multiple of it. It is most common to use the period of the voltage or current waveform, which, depending on the element, may or may not be the same as the fundamental period of the instantaneous power. The average power when the instantaneous power is periodic is, therefore, defined to be

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)dt \ , \tag{4.12}$$

where T is any period and t_0 is an arbitrary reference time. Most commonly, T is selected to be the fundamental period of the voltage or current and t_0 is selected to be zero.

The effective voltage (or current) is a frequently used parameter to describe periodic voltages (or currents). The effective voltage V_{EFF} associated with a periodic voltage $v(t)$ equals that constant voltage that produces the same average power as the periodic voltage when applied to a resistor. Since, for a resistor, $p(t) = v^2(t)/R$,

[1] Recall that a function $x(t)$ is periodic with a period T if $x(t + T) = x(t)$ for all t. The smallest such T is called the fundamental period. Thus, for example, it is easily verified that for a sinusoidal voltage of frequency f (and therefore fundamental period, $1/f$), the instantaneous power absorbed by a resistor is still periodic with period $1/f$, but now has a fundamental period of $1/(2f)$.

this definition implies that

$$\frac{1}{R}V_{EFF}^2 = \frac{1}{T}\int_{t_0}^{t_0+T}\frac{v^2(t)}{R}dt, \tag{4.13}$$

where T is a period and t_0 is any instant of choice. Thus, for a periodic voltage

$$V_{EFF} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} v^2(t)dt} \tag{4.14}$$

Since the effective voltage is the square root of the time-averaged voltage squared, the term root-mean-square or RMS voltage, denoted V_{RMS}, is also commonly used for this quantity, i.e., for a **sinusoidal voltage**,

$$\boldsymbol{V_{RMS} = V_{EFF}}. \tag{4.15}$$

The effective and root-mean-square currents are defined in a similar manner.

A rather trivial but still interesting example in calculating the RMS value of a voltage waveform occurs for a constant voltage. If $v(t) = V_0$, a constant for all time, then it is obvious for any choice of averaging time that the effective or RMS voltage is just V_0 . For a less trivial example, consider $v(t) = A \sin (2\pi ft)$. Selecting $T = 1/f$ and $t_0 = 0$, it is seen that the corresponding effective or RMS voltage is given by

$$V_{RMS} = \sqrt{\frac{1}{T}\int_o^T | A\sin(2\pi ft)|^2 \ dt} \quad . \tag{4.16}$$

Using the double angle formula for the cosine function, this simplifies to

$$V_{RMS} = \sqrt{\frac{1}{2T}A^2\int_0^T[1-\cos(4\pi ft)]dt} = \frac{A}{\sqrt{2}} \tag{4.17}$$

Thus, the effective or RMS voltage for a sinusoidal voltage is approximately 70.7% of the peak voltage. One must be careful when specifying a sinusoidal voltage or, for that matter, any voltage. It is common practice to use the RMS voltage without specifically indicating this fact. For example, the common voltage supplied to residences in North America by power companies is said to be 120 volts at 60 Hertz, and this is the specification seen on light bulbs and household appliances. This is the RMS value of the voltage with the peak voltage being approximately 170 volts. You can verify this by observing the voltage waveform at an electrical outlet with an oscilloscope, but do so with caution!

At this point, it is worth discussing how to use the instruments in our laboratory to measure the RMS voltage for a periodic waveform. When the DMM is set to the DC mode, then the DMM reading will be $\boldsymbol{V_{DC}}$**, the average value or DC component of the signal**. When the DMM is set to the AC mode, then its reading will be $\boldsymbol{V_{AC}}$**, the RMS value of the signal with the DC component removed**.

In order to see how the DMM performs this measurement, one can consider the equivalent circuit shown in Figure 4.2. Essentially, this type of circuit is used in both the DMM and the oscilloscope to control the waveform being measured.

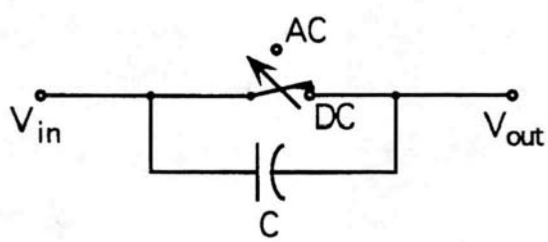

Figure 4.2: The circuit used to perform DC and AC coupling in the oscilloscope and in the DMM.

Consider DMM measurements first. With the switch closed (DC Mode), the entire signal passes through to the DMM. The DMM then averages the signal and, if this signal is periodic, it averages the signal value over one period. With the switch open (AC Mode), V_{in} is connected to V_{out} through C. This is a high-pass filter that blocks low frequencies. Hence, it blocks the DC component of the signal since capacitance looks like an open circuit to DC voltages. Thus, with the switch in the AC Mode, the DMM measures the RMS value of the signal with the DC component removed.

It can be shown that in order to calculate the effective or true RMS value of the input signal, i.e., the RMS value with the DC signal included, one must use the following formula:

$$V_{EFF} = V_{RMS} = \sqrt{V_{DC}{}^2 + V_{AC}{}^2} \qquad (4.18)$$

Consider now the digital oscilloscope. Referring again to Figure 4.2, with the switch closed (DC Coupling), the entire input signal v_{in} passes through to the scope display. However, with the switch open (AC Coupling), v_{in} is connected to the scope display through C. This is a high-pass filter that blocks low frequencies. Hence, it blocks the DC component of the signal since capacitance looks like an open circuit to DC voltages. Thus, with the switch in the AC Coupling position, the scope displays the input signal with the DC component removed.

The final subject of this subsection is a very important result known as Parseval's equation, which relates the effective value of an arbitrary periodic voltage to the effective values of each harmonic in the Fourier series representation of the voltage. As before, let $v(t)$ be an arbitrary periodic voltage with period T. Then, using Fourier series expansion, v(t) can be represented as a linear combination of harmonically related sinusoids according to the formulas presented previously. By using these formulas, the square of the effective or RMS voltage corresponding to $v(t)$ is

$$\begin{aligned} V_{RMS}^2 &= \frac{1}{T}\int_0^T v^2(t)dt \\ &= \frac{1}{T}\int_0^T v(t)\left[a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi f_n t) + \sum_{n=1}^{\infty} b_n \sin(2\pi f_n t)\right]dt \qquad (4.19) \end{aligned}$$

By interchanging the order of integration and summation, we obtain

$$V_{RMS}^2 = a_0 \frac{1}{T}\int_0^T v(t)dt + \sum_{n=1}^{\infty} a_n \frac{1}{T}\int_0^T v(t)\cos(2\pi f t)dt$$

$$+ \sum_{n=1}^{\infty} b_n \frac{1}{T}\int_0^T v(t)\sin(2\pi f t)dt \; . \qquad (4.20)$$

It follows from this equation that

$$V_{RMS}^2 = a_0{}^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n{}^2 + b_n{}^2) \; . \qquad (4.21)$$

Let $A_0 = a_0$, $A_n = a_n/\sqrt{2}$, and $B_n = b_n/\sqrt{2}$, which are the RMS values of the constant and sinusoidal terms in the Fourier series for $v(t)$. We then conclude that

$$V_{RMS}^2 = A_0^2 + \sum_{n=1}^{\infty}(A_n^2 + B_n^2) \; . \qquad (4.22)$$

Thus, the square of the effective or true RMS value of a periodic voltage is the sum of the squares of the RMS values of each component in the Fourier series of that voltage. Equation (4.22) is called ***Parseval's* equation.**

There is yet another useful interpretation of ***Parseval's* equation**. Dividing by R yields

$$\frac{1}{R}V_{RMS}^2 = \frac{1}{R}A_0^2 + \sum_{n=1}^{\infty}(\frac{1}{R}A_n^2 + \frac{1}{R}B_n^2) \; , \qquad (4.23)$$

which may be written as

$$P = \sum_{n=0}^{\infty} P_n \qquad (4.24)$$

where

$$P_0 = \frac{1}{R}A_0^2 \; , \qquad (4.25)$$

and, for $n \geq 1$,

$$P_n = \frac{1}{R} A_n^2 + \frac{1}{R} B_n^2 \; . \tag{4.26}$$

In these equations, P_0 is the average power dissipated by the zero-frequency component of the Fourier series and P_n is the power dissipated by the nth harmonic component. Note that P_n contains the contribution of both the sine and cosine components at frequency $f_n = nf_0 = n/T$. Thus, Parseval's equation implies that the total average power dissipated in the resistor by a periodic voltage is the sum of the average powers dissipated by each harmonic component of the voltage.

4.3.3 Power and Energy for Sinusoidal Waveforms

Now suppose that the voltage waveform $v(t)$ in Figure 4.1 is a sinusoidal function of time. This is an important special case for a number of reasons, the most important of which is the fact that the power line voltage is sinusoidal; hence, knowledge of the behavior of power and energy in this case allows one to analyze such things as the amount of power required from the electric company to operate a particular device. As in the case of periodic waveforms, if the element of Figure 4.1 is linear and time invariant and $v(t)$ is a sinusoidal function of time, then $i(t)$ will also be a sinusoidal function of time. Therefore, we can write these two waveforms as follows:

$$v(t) = V_m \cos(\omega t + \theta_v) \tag{4.27}$$

and

$$i(t) = I_m \cos(\omega t + \theta_i). \tag{4.28}$$

Since we are operating in the sinusoidal stead-state, we can choose any convenient reference for zero time. Engineers have found it convenient when making power calculations to choose zero time to correspond to the instant of time when the *current is passing through a positive maximum*. This reference system requires that we shift both the voltage and current by θ_i. Thus, the above equations for $v(t)$ and $i(t)$ become

$$v(t) = V_m \cos(\omega t + \theta_v - \theta_i) \qquad (4.29$$

$$i(t) = I_m \cos \omega t \;. \qquad (4.30)$$

The instantaneous power $p(t)$ is simply the product of $v(t)$ and $i(t)$, and is given by

$$p(t) = V_m I_m \cos(\omega t + \theta_v - \theta_i) \cos \omega t \;. \qquad (4.31)$$

In order to calculate the average power associated with sinusoidal signals, one can again use Equation (4.12). However, rather than performing this integration directly using Equation (4.31), one can simplify the calculation by first expanding Equation (4.31) using the "cos α cos β" and "cos (α + β)" trigonometric identities, i.e.,

$$\cos(\omega t + \theta_v - \theta_i) \cos \omega t = \frac{1}{2}\cos(\theta_v - \theta_i) + \frac{1}{2}\cos(2\omega t + \theta_v - \theta_i) \qquad (4.32)$$

and

$$\cos(2\omega t + \theta_v - \theta_i) = \cos(\theta_v - \theta_i)\cos 2\omega t - \mathit{sin}\,(\theta_v - \theta_i) \sin 2\omega t \qquad (4.33)$$

After expanding Equation (4.31) using Equation (4.32) and then simplifying the result using Equation (4.33) , the instantaneous power becomes

$$p(t) = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\cos 2\omega t$$
$$- \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)\sin 2\omega t \; . \qquad (4.34)$$

Now, to compute the average power P, all we need to do is use Equation (4.12). To perform this integration, recognize that the integral of either cos $2\omega t$ or sin $2\omega t$ over one period is zero. Using this fact, the integration is trivial, and the average power is given by

$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) \; . \qquad (4.35)$$

Through careful study of Equation (4.34), we can note the following characteristics of the instantaneous power:

(1) The frequency of the instantaneous power is twice the frequency of the voltage or current. This follows directly from the two double frequency terms on the right-hand side of Equation (4.34). From this equation, also note that it is possible for the instantaneous power to be negative for part of each cycle, even if the network between the terminals is passive. Recall that in a network that is completely passive, negative power implies that energy that has been stored in inductors or capacitors is now being extracted.

(2) If the **circuit between the terminals is purely resistive, then the voltage and current will be in phase, i.e.** $\boldsymbol{\theta_v = \theta_i}$. Thus, from Equation (4.34), the instantaneous power is given by

$$p(t) = V_m I_m/2 + V_m I_m/2 \cos(2\omega t). \qquad (4.36)$$

Using Equation (4.35) with $\theta_v = \theta_i$, the instantaneous power becomes

$$p(t) = P + P\cos(2\omega t)\ . \qquad (4.37)$$

This result is often referred to as the instantaneous real power Examining this equation, we see that the instantaneous real power can never be negative, that is, power can never be extracted from a purely resistive circuit. Furthermore, the average power (P) is often referred to as the "real power", which is a term used to describe power that is transformed from electrical to non-electrical form. In the case of a resistor, this means thermal energy or heat.

(3) If the circuit between the terminals is **purely inductive, the voltage will lead the current by +90°, i.e.,** $\boldsymbol{\theta_v = (\theta_i + 90°)}$**.** This means that $\theta_v - \theta_i = +90°$, and Equation (4.34) for the instantaneous power becomes

$$p(t) = -\frac{V_m I_m}{2}\sin 2\omega t \qquad (4.38)$$

Note that in this case, the average power is zero. This means that, in a circuit that is purely inductive, there is no transformation of energy from electrical to non-electrical form. The instantaneous power in such a circuit oscillates between the circuit and the source driving the circuit. When $p(t) > 0$, the energy is being stored in the magnetic fields associated with the inductor, and when p(t) < 0, energy is being extracted from the magnetic fields associated with the inductor.

(4) If the circuit between the terminals is **purely capacitive, the voltage will lag the current by 90°, i.e.,** $\boldsymbol{\theta_v = (\theta_i - 90°)}$**.** This means that $\theta_v - \theta_i = -90°$, and the equation for the instantaneous power reduces to

$$p(t) = \frac{V_m I_m}{2}\sin 2\omega t \qquad (4.39)$$

The average power is again zero in this case, so there is no transformation of energy from electrical to non-electrical form. The energy oscillates between the driving source and the circuit. When $p(t) > 0$, the energy is being stored in the electric fields associated with the capacitor, and when $p(t) < 0$, energy is being extracted from the electric fields associated with the capacitor.

From the above discussion, we are led to the definition of a new term, namely, the reactive power. The reactive power is the power associated with purely inductive or capacitive circuits. The reactive power is denoted by Q and is defined as

$$Q = \frac{V_m I_m}{2} \sin(\theta_v - \theta_i) \,. \qquad (4.40)$$

Using the definitions for P and Q, we can rewrite the expression for instantaneous power as

$$p(t) = P + P \cos 2\omega t - Q \sin 2\omega t \,. \qquad (4.41)$$

Some comments regarding real and reactive power are in order. First, notice that both P and Q have the same dimension. However, in order to distinguish between real and reactive power, we shall use the term "var" for reactive power. The term "**var**" is an acronym for the phrase "**volt-ampere reactive**". Second, note that the way our reference for zero time was chosen, i.e., the way we have written Equations (4.29) and (4.30), leads to Q being positive for inductors and negative for capacitors. In addition, the angle $\theta_v - \theta_i$ is referred to as the power factor angle. Furthermore, the power factor (pf) and the reactive factor (rf) are defined as follows.

$$\text{pf} = \cos(\theta_v - \theta_i) \qquad (4.42)$$

and

$$\text{rf} = \sin(\theta_v - \theta_i) \,. \qquad (4.43)$$

Before completing this subsection, we will introduce the concept of complex power. This concept is very useful when calculating the real and reactive power in circuits operating in the sinusoidal steady-state. The complex power S is defined according to

$$S = P + jQ \tag{4.44}$$

and the magnitude of the complex power, $|S|$, is defined as the apparent power. Dimensionally, S and $|S|$ are the same as the real and reactive power, so in order to distinguish these two quantities from both the real and reactive power, we will use the term "volt amperes" as the units for both complex power and apparent power.

Now, if we substitute the definitions for P and Q into Equation (4.44), we obtain the following equation:

$$S = \frac{V_m I_m}{2}\left[\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)\right]. \tag{4.45}$$

This equation simplifies to

$$S = \frac{V_m I_m}{2} e^{j(\theta_v - \theta_i)} = \frac{1}{2} V_m I_m \underline{/\theta_v - \theta_i} \quad . \tag{4.46}$$

Now recall that $V = V_m \underline{/\theta_v}$ is the phasor representation of $v(t)$, and $I = I_m \underline{/\theta_i}$ is the phasor representation of $i(t)$. Clearly, if we examine Equation (4.46), we see that this equation can be written as

$$S = \frac{1}{2} V I^* . \tag{4.47}$$

That is, the complex power at a pair of terminals is one-half the product of the phasor voltage and the complex conjugate of the phasor current at these terminals. This equation will prove useful when we want to calculate the real, reactive, and complex power at the terminals of an electrical element such as that shown in Figure 4.1.

4.4 Advanced Preparation

The following advanced preparation is strongly recommended before coming to the laboratory:

(1) Thoroughly read and understand the theory and procedures

(2) Compute the theoretical values in the table shown in the Periodic Waveforms Chart. (Note: This chart will be given to you as a handout by your instructor.)

(3) In the circuit shown in Figure 4.4, determine the frequency f required to make $R = X_C$, where X_C is the capacitive reactance given by $X_C = 1/2\pi fC$.

4.5 Experimental Procedure

In this experiment you will be concerned with the measurement of RMS voltages across resistive elements. The basic setup for this experiment is as follows. A resistor R supplied by the instructor is to be connected across either a constant voltage from the DC power supply or a periodic voltage from the HP 33120A function generator, as indicated in Figure 4.3. The temperature of the resistor T_R can be

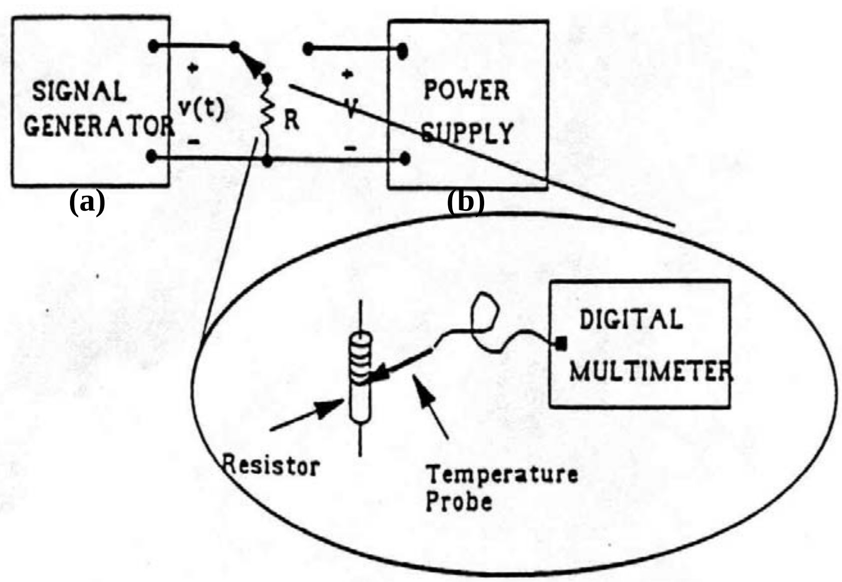

Figure 4.3: Setup for Resistor Power Measurement

measured using a temperature probe connected to a digital multimeter. Some care is required to obtain consistent temperature measurements. Try to place the probe tip at the same orientation and location on the resistor for each measurement. When performing this experiment, be sure to turn the temperature probe off when you finish using it because it is battery powered. Furthermore, ***do not touch the resistor*** when you are applying a voltage to it; ***it may become quite hot*** with some of the voltages in this experiment.

Now complete the following procedures:

4.5.1 For the 1 W, 47Ω resistor, calculate the maximum voltage (V_{max}) allowed by its power rating. Check this value with your instructor. With this resistor connected to the DC power supply output as indicated in Figure 4.3(a), set the power supply level to 0 volt and measure the steady state temperature of the resistor (T_R) in degrees Celsius with the temperature probe. Repeat this measurement for integer voltage settings between 0 volt and V_{max} volts. For best results, take these measurements in order of ascending voltage, allowing the resistor temperature to stabilize at each voltage.

4.5.2 With the resistor connected to the HP 33120A function generator output as indicated in Figure 4.3(b), adjust the function generator to produce each of the first six output waveforms shown on the **Periodic Waveforms Chart (PWC)** at a frequency of f_1. Use DC coupling on Channel 1 and AC coupling on Channel 2 of the oscilloscope. Complete the following for each waveform:

(a) Use the measure function to display Channel 1 frequency and RMS voltage. Also display the RMS voltage for Channel 2. Make a hardcopy of the display. Copy the Channel 2 waveform on the PWC.

(b) Measure V_{DC} and V_{AC} across the resistor using the DMM and calculate V_{RMS}, the effective or true RMS voltage. Record these on the PWC.

(c) Record the steady-state temperature of the resistor T_R on the PWC.

4.5.3 Repeat steps (b) and (c) in Section 4.5.2 above using the waveform specified by your instructor for frequencies $f_2 = 0.5 f_1$, $f_3 = 2f_1$, and $f_4 = 5f_1$.

4.5.4 Now add a capacitor of value C in series with the resistor R as shown in Figure 4.4. Use the HP 33120A function generator as the voltage source $v(t)$.

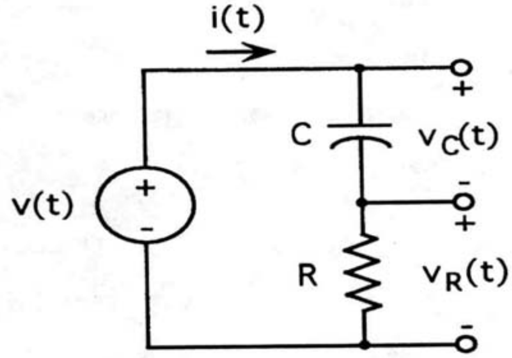

Figure 4.4: Series RC Circuit

Complete the following for this circuit:

(a) Determine the frequency f required to make $|X_C| = R$, where X_C is the capacitive reactance and is given by $X_C = -1/(2\pi fC)$. Note that if you came to lab adequately prepared, you should have already completed this part!

(b) Set the function generator to produce a sine wave with a peak-peak voltage of V_2, a DC offset of V_3, and the frequency you calculated in 4.5.4 (a) above. Using AC coupling for both signals, make a copy of the voltage across the resistor compared to the function generator voltage as illustrated in Figure 4.5, showing the magnitudes of $v(t)$ and $v_R(t)$, the period T, and the phase angle φ.

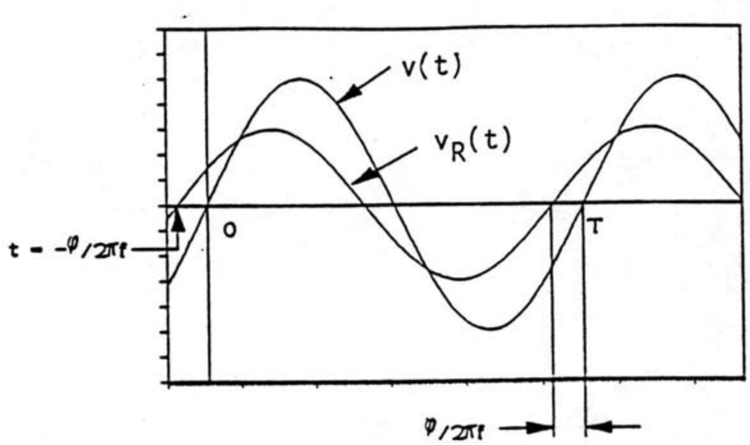

Figure 4.5: Oscilloscope Measurements

(c) Measure V_{DC} and V_{AC} across the resistor and the resistor temperature using the DMM.

4.5.5 Construct the circuit shown in Figure 4.6. Apply a sine wave

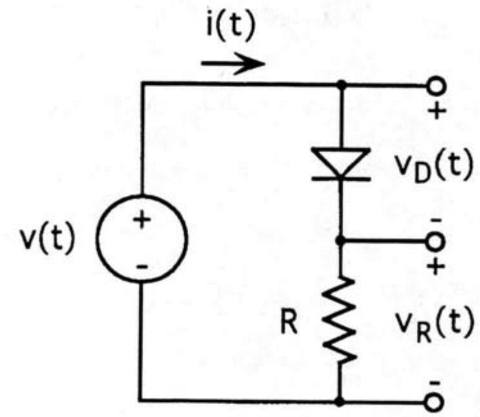

Figure 4.6: Diode Circuit

at a frequency of f_5 from the function generator in order to produce the half wave rectified voltage across the resistor as shown as the last waveform at the bottom of the Periodic Waveforms Chart. Now repeat steps (a), (b), and (c) as defined in Section 4.5.2 for this waveform.

4.6 Report

4.6.1 Construct a table listing the resistor temperature T_R and the DC voltage data that you took in Step 4.5.1. Add a column that shows the calculated power absorbed by the resistor based on the measured value of the resistor.

4.6.2 Plot the temperature versus voltage points from your tabulated data of Section 4.6.1 above and sketch the "best-fit" nonlinear curve based on your data points. This graph represents a ***Calibration Curve*** for your resistor and provides the temperature as a function of the DC voltage across the resistor. This DC voltage is by definition V_{EFF}, the *Effective (True RMS)* voltage.

4.6.3 Plot the temperature versus average power points from your data from Section 4.5.1. Develop a linear equation for is the best fit for your data using order 1 for the polynomial, i.e., $y = a_0 + a_1x$. Use a computer program or do this manually. See Appendix I in Section 4.8 of this experiment. Add a plot of this function to the temperature versus average power points plot to develop a calibration curve for the temperature versus power absorbed by your resistor.

4.6.4 Calculate the Effective (True RMS) voltage values using Equation (4.14) for each of the periodic waveforms on the Periodic Waveforms Chart (PWC) as specified by your instructor. Include detailed calculations in your report.

4.6.5 Calculate the first 15 Fourier series coefficients for the periodic waveforms specified by your instructor. Include detailed calculations in your report.

4.6.6 Use Parseval's equation with 3, then 5, and finally 7 nonzero harmonic coefficients ($n \geq 2$) to predict the *Effective (True RMS) Volta*ge and the resulting average power for each of the waveforms on Periodic Waveforms Chart as specified by your instructor. Also, calculate % Error for each of the three values of V_{EFF} to compare your predictions with the corresponding theoretical average power obtained in 4.6.4 above.

4.6.7 Present your PWC showing the data obtained in Sections 4.5.2 and 4.5.5. Be sure it includes the following for each waveform: the AC coupled waveform as well as the theoretical (T) and measured (M) values of V_{DC}, V_{AC}, V_{EFF} and T_R. Note that the theoretical T_R is obtained from the ***Calibration Curve*** using the theoretical $V_{\mathrm{EFF.}}$

4.6.8 Based on your data from Section 4.5.2 for each of the periodic waveforms, construct a table that indicates V_{EFF}, the *Effective (True RMS) Voltage*, obtained from the following: (1) the theoretical V_{EFF}, (2) V_{EFF} obtained using the DMM measurements, (3) the RMS voltage shown on Channel 1 on the scope and (4) the resistor calibration curve based on measured temperature.

4.6.9 Construct a table similar to the one described in Section 4.6.8 that compares the average power absorbed by the resistor using your (1) temperature versus power curve, (2) V_{EFF} calculated from the DMM measurements, (3) V_{EFF} from your theoretical calculations, and (4) the RMS voltage shown on Channel 1.

4.6.10 Determine the percent error in the measured results presented in the above two tables, assuming that calculated values are correct. In other words, for each waveform, you should calculate percent error for the following four quantities: (1) the RMS voltage obtained from the temperature versus voltage curve, (2) the RMS voltage obtained from the DMM measurements, (3) the average power obtained from the temperature versus power curve, and (4) the average power obtained from the DMM measurements. Comment on the results. How well do your results agree with the calculated values? Identify those variables that could introduce errors into your measurements.

4.6.11 Present the data that you took in Section 4.5.2 and 4.5.3 in tabular form. For each frequency, give the following information: resistor temperature (T_R), V_{EFF} determined from both the DMM readings and the temperature versus voltage curve, and the average power determined from both the DMM readings and the temperature versus power curve. Determine if the RMS voltage and average power change as a function of frequency.

4.6.12 Consider the circuit shown in Figure 4.4. Since the circuit is linear and time invariant, recall that it follows that a signal generator voltage of the form $v(t) = A \sin(2\pi ft)$ will cause a current of the form $i(t) = B \sin(2\pi ft + \varphi)$, where the magnitude B and the phase angle φ depend on R, C, A, and f. Analyze the circuit to determine B and φ in terms of R, C, A, and f. Also, show mathematically that the average power absorbed by the RC series circuit is $(AB/2)\cos\varphi$. You *may not* use Equation (4.35) to obtain this result. (Hint: begin with Equation (4.12) and perform the integration.

4.6.13 Show your calculations for determining in Section 4.5.4 (a) the frequency f that was required to make $|X_C| = R$, where $X_C = -1/(2\pi fC)$. Assuming a sine wave $v(t)$ with $A = V_2/2$, what will be the values of B in amps and φ in degrees for this value of f? Finally what will be the value of $v_R(t)$?

4.6.14 Present the copy of the voltage $v_R(t)$ across the resistor compared to the function generator voltage $v(t)$ that you made in Section 4.5.4 (a). Compare your measurements with the results of the calculations performed in Sections 4.6.12 and 4.6.13.

4.6.15 Present your measurements for T_R, V_{DC}, and V_{AC} that you took in Section 4.5.4. Determine the average power absorbed by the resistor from (1) your calibration curve and (2) your DMM measurements. Compare these to the calculated value derived in Section 4.6.12above.

4.6.16 Using the values of B and φ that you calculated in 4.6.13 above, answer the following questions:

(a) What is the peak value of the instantaneous power delivered by the source $v(t)$?

(b) What is the complex power ?

(c) What is the reactive power ?

(d) What is the power factor and the reactive factor of the RC load ?

4.6.17 Present your measurements for the temperature of the resistor, V_{DC}, and V_{AC} that you took in Part 4.5.5. Determine the RMS voltage and average power absorbed by the resistor using (1) your calibration curve and (2) your DMM measurements, and compare these values to the calculated values using the formulae for the RMS voltage determined in Section 4.6.7 on the PWC.

4.7 References

1. Nilsson, J. W., **Electric Circuits**, (6th ed.), Prentice Hall, Upper Saddle River, New Jersey 2001 (see Chapter 9 for sinusoidal steady-state analysis, Chapter 10 for sinusoidalsteady-state power calculations, and Chapter 16 for Fourier series).

2. Straight, P., **Probability and Statistics with Applications** (see Chapter 14, pages 438-449 for polynomial fitting).

4.8 Appendix I: Fitting Polynomials to Experimental Data

Suppose that $\{x_i, y_i; i = 1, 2, 3, ..., n$) represents n measured data points. For this experiment, x_i is a temperature value, y_i is the corresponding average power value, and the ith measurement of these quantities corresponds to the results obtained with a particular setting of the parameters of the function generator. A plot of temperature -power data pairs will have a scatter diagram that might look like the data points shown in Figure 4.7. These points may somewhat follow along some smooth function, but there will be deviations due to various sources of measurement error, including imperfect readings of oscilloscope traces, quantization errors in digital multimeters, and fundamental instrument limitations.

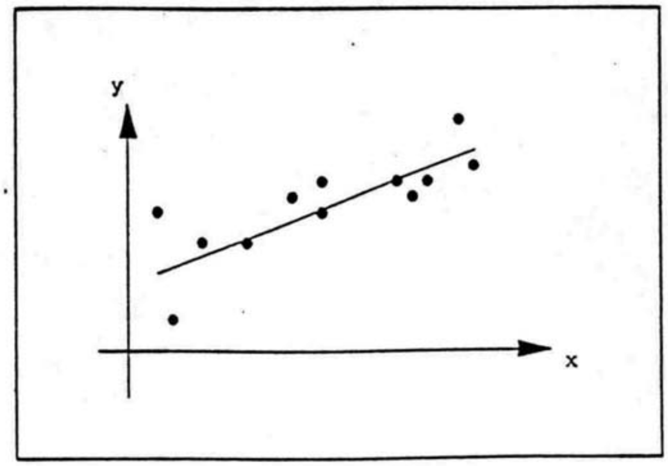

Figure 4.7: Fitting a Polynomial to Data

It is often desired to fit a simple function to data in order to have an equation that describes them. Polynomials, sums of exponentials, and sums of sinusoids are examples of functions that might be used for the fit. Here we examine the use of polynomials. Thus, let $y(x)$ be a polynomial of degree m,

$$y(x) = a_0 + a_1x + a_2x^2 + \cdots\cdots + a_mx^m \ . \qquad (4.48)$$

A graph of such a polynomial for m = 1 (i. e., a straight line) is shown superimposed on the data in Figure 4.7.

The objective in fitting a polynomial to some given data is to select the degree of the polynomial m and the polynomial coefficients $\{a_0,\ a_1,\ \cdots\ ,\ a_m\ )$ so that the polynomial is close in some sense to all the data points. A common approach is to'eyeball' the data to select the polynomial degree; if the data seem to follow a straight line closely, then m = 1 would be selected; if the data depart somewhatfrom a straight line, then perhaps m = 2 would be selected, etc. Keeping the degreelow is desirable because this keeps the number of coefficients to be determined low,but if the degree is too low, the fit may not be accurate enough. For a given degree, the coefficients are selected to minimize some measure of the errors, with the mean square error being a common choice. This mean square error is defined by

$$\langle e^2(a_0, a_1, ..., a_m)\rangle = \frac{1}{n}\sum_{i=1}^{n}\left[y_i - y(x_i)\right]^2 \ , \tag{4.49}$$

where

$$y(x_i) = a_0 + a_1 x_i + a_2 x_i^2 + \cdots + a_m x_i^m \tag{4.50}$$

is the polynomial evaluated at $x = x_i$. By taking the derivatives of the mean square error with respect to each of the polynomial coefficients and setting the results to zero, the following set of linear equations for minimizing the mean square error is obtained

$$a_0 + < x > a_1 + < x^2 > a_2 + ... + < x^m > a_m = < y > \tag{4.51}$$

$$< x > a_0 + < x^2 > a_1 + < x^3 > a_2 + ... + < x^{m+1} > a_m = < yx > \tag{4.52}$$

$$< x^m > a_0 + < x^{m=1} > a_1 + < x^{m+2} > a_2 + ... + < x^{2m} a_m = < yx^m > \tag{4.53}$$

where

$$< x^k > = \frac{1}{n}\sum_{i=1}^{n} x_i^k \ , \tag{4.54}$$

and

$$< yx^k > = \frac{1}{n}\sum_{i=1}^{n} y_i x_i^k \ . \tag{4.55}$$

These last two quantities are called the moments of the data. The procedure, then, is to calculate the moments of the data using the measured values and then to solve these linear, algebraic equations for the coefficients of the best-fit polynomial.

As an example, for a straight line fit, m = 1, and $y(x) = a_0 + a_1 x$. The coefficients for the best straight line fit satisfy the linear equations:

$$a_0 + < x > a_1 = < y > \tag{4.56}$$

$$< \mathrm{x} > a_0 + < \mathrm{x}^2 > a_1 = < yx > \ . \tag{4.57}$$

The square root of the minimum value of the mean square error that results is called the minimum RMS error.