Projection Theorem

Instructor: Yaqi Chen

Department of Electrical & Systems Engineering,
Washington University in St. Louis,
Saint Louis, MO, 63130, U.S.A.
Jolley Hall, Room 426
Mobile: 0001-314-603-6611
Email: chen.y@ese.wustl.edu
Case Study

\[ A = \begin{bmatrix} 5,1 \\ 3,2 \\ 1,4 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \]

\[
5x + y = 1 \\
3x + 2y = 3 \\
x + 4y = 2
\]

No Solution!

Least Squares Estimation

\[
m \text{ in } \|Ax - b\|
\]
Linear Algebra Review

Orthogonal Complement

Given a set $S \subseteq X$, its orthogonal complement is

$$S^\perp = \{ x \in X | \langle x, R \rangle \geq 0, \forall R \in S \}$$

Subspace

A nonempty set $S \subseteq V$ is called a subspace of $V$, if $\alpha x + \beta y \in S$, $\forall x, y \in S$

Any subspace contains zero vector.
Linear Algebra Review

Range Space and Null Space

\[ R(A) = \{ x \mid Az = x \} \]

\[ N(A) = \{ x \mid Ax = 0 \} \]

\[ N(A^T) = R(A)^\perp \]

\[ A \in \mathbb{R}^{m \times n}, \text{ then } N(A) \text{ is a subspace of } \mathbb{R}^n, \]

\[ R(A) \text{ is a subspace of } \mathbb{R}^m. \]
Projection Theorem

Let $H$ be a Hilbert space and $M$ a closed subspace of $H$. Corresponding to any vector $x \in H$, there is a unique vector $m_0 \in M$ such that $\|x - m_0\| \leq \|x - m\|$ for all $m \in M$.

Furthermore, a necessary and sufficient condition that $m_0 \in M$ be the unique minimizing vector is that $x - m_0$ be orthogonal to $M$.

Proof can be found in Optimization by Vector Space Methods by David G. Luenberger Page 50-52.
Projection Theorem

Three-dimensional Version of Projection Theorem
Applications

Finite –dimensional Version

Ax = b , x ∈ R^n , A ∈ R^{m×n} , m > n and A has full column rank, i.e. rank ( A ) = n . Goal:

min ||Ax − b||

Solution:

R( A ) = M , b → x , according to Projection Theorem,
m_0 ∈ R( A ) and b − m_0 ⊥ R( A ) . Then b − m_0 ∈ R( A )^⊥

b − m_0 ∈ N( A^T ) , then A^T ( b − m_0 ) = 0 .
Applications

Finite –dimensional Version

Continued:

\[ A^T (b - m_0) = 0 \], then \[ A^T b = A^T A x_0 \].

Since \( A \) has full column rank, \( A^T A \) is invertible.

\[ x_0 = (A^T A)^{-1} A^T b \]
Applications

Homework 2-13

Find the vector $\hat{x} \in \mathbb{R}^N$ that minimizes

$$(x - \hat{x})^T (x - \hat{x})$$

under the constraint that $A^T \hat{x} = 0$, with $A^T$ a $p \times N$ matrix with rank $p \leq N$.

Solution:

$\hat{x} \in N(A^T) \Rightarrow \hat{x} \in R(A)^\perp \Rightarrow x - \hat{x} \in R(A) \Rightarrow Az = x - \hat{x}$

$\Rightarrow A^T Az = A^T x - A^T \hat{x} \Rightarrow A^T Az = A^T x \Rightarrow z = (A^T A)^{-1} A^T x$

$\Rightarrow \hat{x} = x - Az = x - A(A^T A)^{-1} A^T x = (I - A(A^T A)^{-1} A^T) x$
Restatement of Projection Theorem

Let $H$ be a Hilbert space and $M$ a closed subspace of $H$. Let $x$ be a fixed element in $H$ and let $V$ be the linear variety $x + M$. Then there is a unique vector $x_0$ in $V$ of minimum norm. Furthermore, $x_0$ is orthogonal to $M$.
Applications

Minimum Norm Problem

\[ Ax = b, \ x \in \mathbb{R}^n, \ A \in \mathbb{R}^{m \times n}, \ m < n \] and \( A \) has full row rank, i.e. \( \text{rank}(A) = m \). (Infinite Solutions)

Goal: \( \min \|x\| \) invertible \( AA^T \)

Solution:

Solution set \( \{ x_0 + N(A) \} = V \) is a linear variety.

\( \hat{x} \perp N(A) \Rightarrow \hat{x} \in N(A)^\perp = R(A^T) \Rightarrow A^T z = \hat{x} \)

\( \Rightarrow A\hat{x} = b = AA^T z \Rightarrow z = (AA^T)^{-1} b \Rightarrow \hat{x} = A^T (AA^T)^{-1} b \)
Approximation Theory

\[ \{y_1, y_2, \ldots, y_n\} \in H \quad M = \text{span}\{y_i\} \]

Goal: \( \forall x \in H \), find \( \tilde{x} \in M \) s.t. \( \min \|x - \tilde{x}\| \).

Solution:

\[ \tilde{x} = \sum_{i=1}^{n} \beta_i y_i, \quad \beta = (G^T)^{-1} b, \quad b = [c_1, \ldots, c_n]^T, \]

\[ \langle x, y_i \rangle = c_i, \quad \beta = [\beta_1, \ldots, \beta_n], \quad G_{ij} = \langle y_i, y_j \rangle. \]

G is the Gram Matrix.
Applications

Minimum Norm in a Hilbert Space

\( \{ y_1, y_2, \ldots, y_n \} \in H \) are linearly independent.

\( c_i, i = 1, \ldots, n \) are fixed constants.

Goal: \( \min \| x \|, \) s.t. \( \langle x, y_i \rangle = c_i. \)

Solution:

\[
\hat{x} = \sum_{i=1}^{n} \beta_i y_i, \beta = (G^T)^{-1} b, b = [c_1, \ldots, c_n]^T.
\]

Solution set \( \{ x_0 + M^\perp \} = V \) is a linear variety, where

\[
M = \text{span}\{ y_i \}, \hat{x} \perp M^\perp \implies \hat{x} \in M \implies \hat{x} = \sum_{i=1}^{n} \beta_i y_i.
\]
<table>
<thead>
<tr>
<th>Finite-Dimensional Version of Project Thm</th>
<th>Minimum Norm Problem By Project Thm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Ax = b, A \in \mathbb{R}^{m \times n}, m &gt; n$</td>
<td>$Ax = b, A \in \mathbb{R}^{m \times n}, m &lt; n$</td>
</tr>
<tr>
<td>$\text{rank}(A) = n, \min |Ax - b|$</td>
<td>$\text{rank}(A) = m, \min |x|$</td>
</tr>
<tr>
<td><strong>Solution:</strong></td>
<td><strong>Solution:</strong></td>
</tr>
<tr>
<td>$x_0 = (A^T A)^{-1} A^T b$</td>
<td>$\hat{x} = A^T (AA^T)^{-1} b$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Approximation Theory</th>
<th>Min Norm in H Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>${y_1, y_2, \ldots, y_n} \in H, M = \text{span}{y_i}$</td>
<td>${y_1, y_2, \ldots, y_n} \in H, c_i, i = 1, \ldots, n$</td>
</tr>
<tr>
<td><strong>Goal:</strong></td>
<td><strong>Goal:</strong></td>
</tr>
<tr>
<td>$\forall x \in H, \hat{x} \in M, \min |x - \hat{x}|$</td>
<td>$\min |x|, \text{s.t. } \langle x, y_i \rangle = c_i$</td>
</tr>
<tr>
<td><strong>Solution:</strong></td>
<td><strong>Solution:</strong></td>
</tr>
<tr>
<td>$\hat{x} = \sum_{i=1}^{n} \beta_i y_i, \beta = G^{-1} b, b = [c_1, \ldots, c_n]^T$, $\beta = [\beta_1, \ldots, \beta_n]$, $\langle x, y_i \rangle = c_i, G_{ij} = \langle y_i, y_j \rangle$.</td>
<td>$\hat{x} = \sum_{i=1}^{n} \beta_i y_i, \beta = (G^T)^{-1} b, G_{ij} = \langle y_i, y_j \rangle$, $\beta = [\beta_1, \ldots, \beta_n], b = [c_1, \ldots, c_n]^T$.</td>
</tr>
</tbody>
</table>