Overview of Beamforming

Arye Nehorai

Preston M. Green Department of Electrical and Systems Engineering
Washington University in St. Louis

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Outline

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- Frequency response
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Introduction

- Beamforming: A method used in array signal processing mainly for the following two goals:
  - Finding the direction of a desired signal.
  - Enhancing the desired signal.

- Beamforming is a general filtering operation that combines temporal and spatial filtering.

- It uses a weighted sum of the outputs of multiple sensors at certain time instants.

- In other words, beamforming is a linear combination of the temporal outputs of multiple sensors.
Applications

- Radar
- Sonar
- Biomedicine
- Communications
- Imaging
- Geophysics
- Astrophysics
Spatial vs. Temporal Filtering

- Spatial filtering allows resolving signals sharing the same temporal frequency but spatially separated.

- Temporal filtering can only resolve signals having different temporal frequency content.

- Spatial filtering weights outputs of multiple sensors located at different physical locations.

- Temporal filtering weights the delayed versions of the output of a single sensor.
Spatial Filtering

Spatial Beamformer.

A spatial beamformer weights the outputs of the sensors at a certain time.
Spatial Filtering (Cont.)

Output of the filter is:

\[ y(t) = \sum_{i=1}^{J} w_i^* x_i(t), \]

where \( J \) denotes the number of sensors, \( x_i(t) \)'s are the signals arriving at these sensors at time \( t \), and \( w_i \)'s are the spatial weights. In vector form:

\[ y(t) = w^H x(t), \]

where \( x(t) = [x_1(t), x_2(t), \ldots, x_J(t)]^T \), \( w = [w_1, w_2, \ldots, w_J]^H \), and we have used "\( ^T \)" for transpose and "\( ^H \)" for Hermitian conjugate.
Spatial Filtering Combined With Temporal Filtering

Here, we weight the outputs of the sensors and their delayed versions.
Spatial Filtering Combined With Temporal Filtering (Cont.)

\[ y(t) = \sum_{i=1}^{J} \sum_{p=0}^{K-1} w^*_{i,p} x_i(t - pT), \]

where \( K - 1 \) is the number of delays in each of the sensor channels and \( T \) is the duration of a single delay.

In vector form:

\[ y(t) = \mathbf{w}^H \mathbf{x}(t), \]

where

\[
\begin{align*}
\mathbf{x}(t) &= [x_1(t), x_1(t - T), \ldots, \\
& \quad x_1(t - (K - 1)T), \ldots, x_2(t), \ldots, x_J(t - (K - 1)T)]^T, \\
\mathbf{w} &= [w_{1,0}, w_{1,1}, \ldots, w_{1,(K-1)}, \ldots, w_{2,0}, \ldots, w_{J,(K-1)}]^H.
\end{align*}
\]
• We now analyze the filters in the frequency domain.

• Frequency response of a filter $= \frac{\text{Response to an exponential}}{\text{Exponential}}$

• Frequency response of a spatial filter: For an exponential input $e^{j\omega t}$, where $\omega$ represents the frequency of the exponential, the output is:

$$y(t) = \sum_{i=1}^{J} w_i^* e^{j\omega(t - \Delta_i(\theta))},$$

where $\Delta_i(\theta)$ is the delay due to propagation at each sensor and $\theta$ is the direction of arrival of the signal.
Therefore,

\[
r(\theta, \omega) = \frac{y(t)}{e^{j\omega t}} = \sum_{i=1}^{J} w_i^* e^{j\omega \Delta_i(\theta)}.
\]

In vector form:

\[
r(\theta, \omega) = w^H d(\theta, \omega),
\]

where \( d(\theta, \omega) = [1, e^{-j\omega \Delta_2(\theta)}, \ldots, e^{-j\omega \Delta_J(\theta)}]^T. \)

\( d(\theta, \omega) \) gives a complete idea of the array spatial response. It is often called steering vector, array response vector, or direction vector.
Frequency Response (Cont.)

Example: Uniform Linear Array where \( \delta = d/v \) with \( d \) the distance between the sensors and \( v \) is the waveform speed of propagation.
Frequency Response (Cont.)

- Frequency response of spatial-temporal filter:
  In this case, we also use the delayed versions of the exponential, so the output is:

\[
y(t) = \sum_{i=1}^{J} \sum_{p=0}^{K-1} w_{i,p}^* e^{-j\omega(\Delta_i(\theta) + pT - t)}.
\]

Therefore,

\[
r(\theta, \omega) = \sum_{i=1}^{J} \sum_{p=0}^{K-1} w_{i,p}^* e^{-j\omega(\Delta_i(\theta) + pT)}.
\]

In vector form:

\[
r(\theta, \omega) = w^H d(\theta, \omega),
\]

\[
d(\theta, \omega) = [1, e^{T_{1,1}(\theta)}, \ldots, e^{T_{J,K}(\theta)}],
\]

\[
T_{i,p}(\theta) = -j\omega(\Delta_i(\theta) + pT).
\]
Design Methods

- When designing a beamformer, we first set our objective, which may be one of: signal enhancement, direction finding, interference suppression, and possibly others.

- Then, we choose a mathematical criterion to help us achieve this objective.

- The weights are selected to meet the criteria as close as possible.

- There are two main criteria for designing beamformers:
  - Classical beamforming (data independent) and
  - Adaptive beamforming.
Design Methods: Data Independent

Data-independent methods assume apriori knowledge of second-order statistics of the data.

1. Least-squares error optimization:

- In this method, we choose our filter coefficients to approximate a desired frequency response.

- We want to minimize the error between $r_d(\theta, \omega)$, the desired response, and $r(\theta, \omega)$ that we obtain as a result of our design at $P$ points.
Design Methods: Data Independent (Cont.)

- Mathematically, the optimum weights are given by:

\[ \min_w |A^H w - r_d|^2, \]

where

\[ A = [d(\theta_1, \omega_1), d(\theta_2, \omega_2), \ldots, d(\theta_P, \omega_P)], \]

and

\[ r_d = [r_d(\theta_1, \omega_1), r_d(\theta_2, \omega_2), \ldots, r_d(\theta_P, \omega_P)]^H. \]

- Solution to this problem is the familiar pseudo-inverse:

\[ w_{opt} = (AA^H)^{-1} A r_d. \]
Design Methods: Data Independent (Cont.)

- How to choose the desired response? We can choose it to be unity in the direction of the signal, and zero in the directions of interferences.

- The disadvantage of this method is the requirement of knowing the desired signal and interference directions.

- Also we face the classical problem of the inverses, amplification of the noise when the matrix $A$ is ill-conditioned.
2. Maximum signal-to-noise ratio (SNR):

- In this case we design our beamformer to improve the SNR for a source with a given direction and frequency.

- We maximize the ratio of the SNRs at the output and input.

- Assuming stationarity and a narrow-band random signal, the output power is:

$$ E\{|y(t)|^2\} = |w^H d(\theta_0, \omega_0)|^2 \sigma_s^2 + w^H \Sigma w, $$

where $\sigma_s^2$ is the variance of the signal with direction $\theta_0$ and frequency $\omega_0$, and $\Sigma$ is the covariance matrix of the interferences and noise.
Beamforming

**Design Methods: Data Independent (Cont.)**

- Hence the optimum weights are given by:

\[
\max_w \frac{\text{SNR}_{\text{out}}}{\text{SNR}_{\text{in}}} = \max_w \frac{|w^H d(\theta_0, \omega_0)|^2 \text{trace}(\Sigma)}{w^H \Sigma w J}.
\]

- Solution to this problem is similar to a matched filter:

\[
w_{\text{opt}} \propto \Sigma^{-1} d(\theta_0, \omega_0).
\]

- This method can be used to enhance of the desired signal.

- Disadvantage of this method is the requirement of knowing the frequency and direction of the signal as well as the covariance of interferences and noise.
3. Generalized sidelobe canceller:

- We extract the main signal component from the signals arriving at the sensors using a set of coefficients denoted by $w_m$ in the figure, and eliminate the auxiliary components using a set of coefficients denoted by $w_a$ in the figure.
Design Methods: Data Independent (Cont.)

- To completely cancel the interferences is sometimes impossible and usually results in large white-noise gain.

- Weights are chosen to trade-off interference suppression for white-noise gain.

- Especially useful when the desired signal is weak compared with interference.
4. Linearly-constrained minimum variance beamformer (LCMV):

- In this method, we design our beamformer so that the output power is minimum subject to a constraint.

- The constraint allows us to have a certain phase and amplitude of the response at certain directions.

- Minimizing the output power is necessary to attenuate the contributions from the interferences and noise.
Design Methods: Data Independent (Cont.)

- Mathematically, we can write:

\[ w_{opt} = \arg \min_w (w^H R w), \]

subject to

\[ C^H(\theta, \omega) w = f, \]

where \( R \) is the covariance matrix of the measurements at the sensors, \( C(\theta, \omega) \) is a matrix containing the steering vectors of directions that we want to put constraints on, and \( f \) is a pre-determined complex vector chosen to give the desired amplitude and phase response at the given directions.
Design Methods: Data Independent (Cont.)

• The solution to this minimization problem can be found using Lagrange multipliers, thus:

\[ w_{\text{opt}} = R^{-1}[C^H R^{-1} C]^{-1} f. \]
5. Minimum variance distortionless response (MVDR) or Capon method:

- This is a special case of LCMV with $f = 1$ and $C = d^H(\theta, \omega)$.

- It deserves special attention since it is very effective, robust and commonly used.

- Optimum weights are given by:

$$w_{opt} = \frac{R^{-1}d(\theta, \omega)}{d^H(\theta, \omega)R^{-1}d(\theta, \omega)}.$$

- With these optimum weights, the minimum output power is:

$$E\{y(t)^2\} = w_{opt}^HRw_{opt} = \frac{1}{d^H(\theta, \omega)R^{-1}d(\theta, \omega)}.$$
Design Methods: Adaptive

- All of the above methods require the knowledge of second-order statistics of the data ($R$).

- If we consider that these are known to the designer and we do not need to estimate them, optimum weights are independent of data, hence the name.

- But if we need to estimate the second-order statistics, our optimum weights are now dependent on the data.
Design Methods: Adaptive (Cont.)

- There are two approaches to estimating second-order statistics:
  - Block adaptation: Statistics are estimated from a temporal block of array data, and used in calculating the optimum weights.
  - Continuous adaptation: Estimator for the statistics is adaptive, hence updated as new data arrives. In this case, optimum weights are updated with the new data and converges to the optimum solution.

- An example of block adaptation is the sample covariance matrix:

\[
\hat{R} = \frac{1}{L} \sum_{k=0}^{L} x(k)x^H(k),
\]

where \( L \) is the total number of samples we have.
Design Methods: Adaptive (Cont.)

Example of continuous adaptation:

\[ y_d(t) \rightarrow \hat{w} \rightarrow y(t) \quad u(t) \]

- Here, we use the new arriving data to update our estimate of the second-order statistics.

- We select weights to minimize the mean-squared error:

\[
E\{|y_d(t) - w^H u(t)|^2\} = E\{|y_d|^2(t)\} - w^H E\{u(t) y_d^*(t)\} - E\{u^H(t) y_d^*(t)\} w + w^H R w,
\]

where \( u(t) \) is the input and \( y_d(t) \) the desired output.
Design Methods: Adaptive (Cont.)

- Assuming stationarity, solving this minimization problem we obtain the optimum weights as:
  \[ w_{opt} = R^{-1}E\{u(t)y_d(t)\}. \]

- We can use sample covariance for \( R \) and sample average for \( E\{u(t)y_d(t)\} \), using the arriving data at the sensors, and possibly adapting the new data to get better estimates.

- We can see that this method is similar to multiple sidelobe canceller if we treat the main channel component as \( y_d(t) \) and the auxiliary components as \( u(t) \).

- Also this method is equivalent to the least-squares error problem if we treat the desired signal as \( y_d(t) \) and signals arriving at the sensors as \( u(t) \).
Practical Applications: Numerical Examples

• Now, we demonstrate how beamforming can be used in practical applications by giving a number of numerical examples for:

  – Signal enhancement.
  – Direction finding.
  – Interference suppression.
Signal Enhancement

- We want to eliminate noise to improve performance.

- We can use any of the methods discussed to enhance the desired signal. Here, we give an example of the Capon method for signal enhancement.

- We can enhance the desired signal using Capon beamformer, assuming that we know the direction of the signal, and the covariance matrix of the data arriving at the sensors.

- However, we can use time averages to obtain the covariance matrix assuming ergodicity, when the true covariance is not available.
**Signal Enhancement: Example-1**

- 30 dB signal, additive white Gaussian noise, at each sensor.
- Uniform linear array formed by 6 sensors separated by $\frac{\lambda}{2}$, where $\lambda$ is the wavelength.
- Azimuth of arrival: $\theta = \frac{\pi}{6}$.
Signal Enhancement: Experiment-2

- 20 dB signal, additive white Gaussian noise, at each sensor.
- Uniform linear array formed by 6 sensors separated by $\lambda/2$.
- Azimuth of arrival: $\theta = \pi/6$. 

![Graphs showing signal enhancement](image-url)
Direction Finding

• Direction of arrival (DOA) estimation is a major subject in array signal processing, since it has plenty of applications in radar, communications, and many other areas.

• To enhance a signal using Capon beamformer, we need to know its direction.

• However, we can use Capon beamformer to estimate the direction of the signal, assuming that the covariance matrix of the array measurements are known.

• We scan all possible directions. Estimated direction of the signal is the one that maximizes the output power:

\[
\frac{1}{d^H(\theta, \omega)R^{-1}d(\theta, \omega)}.
\]
Direction Finding: Example

- 30 dB signal, additive white noise, at each sensor.
- Uniform linear array formed by 6 sensors separated by $\lambda/2$.
- Azimuth of arrival: $\theta = \pi/3$. 
Direction Finding: Example

- DOA is successfully estimated as $\pi/3$. 
Interference Suppression

- The LCMV beamformer can be used, for interference suppression as follows.

- Form $C$ using the steering vectors of the signal direction $\theta_0$ and interference direction $\phi$. Use $f$ with unity for the signal, and zero for the interference.

- We can summarize this mathematically as follows:

\[
C = [d(\theta_0, \omega), d(\phi, \omega)],
\]

\[
f = [1, 0]^T.
\]
Interference Suppression: Example-1

- 30 dB signal at azimuth $\theta_0 = \pi/3$, additive white Gaussian noise, at each sensor.

- 30 dB interference at azimuth $\phi = \pi/6$ radians.

- Uniform linear array formed by 6 sensors separated by $\lambda/2$. 
Interference Suppression: Example-1 (Cont.)

- Desired signal is approximately obtained at the output of the beamformer, and interference is eliminated.
Interference Suppression: Example-2

- 30 dB signal at azimuth $\theta_0 = \pi/3$, additive white Gaussian noise, at each sensor.
- 50 dB interference at azimuth $\phi = \pi/6$ radians.
- Uniform linear array formed by 6 sensors separated by $\lambda/2$. 
Interference Suppression: Example-1 (Cont.)

- Desired signal is approximately obtained at the output of the beamformer, and interference is eliminated though it is much stronger than the desired signal.
Performance Analysis

When evaluating a beamformers performance, we can use the cost functions we have mentioned as the criteria. Some performance criteria can be listed as follows:

- Mean-squared error between the desired output and actual output.
- Output signal-to-noise ratio.

Computation time is also a factor.
Design of Beamformer in Absence of Dielectric Wall

The output of the beamformer for a single target is given by:

\[ z_q(t) = \sum_{n=1}^{N} w_n a(x_p) s(t - \tau_n - \tilde{\tau}_n) \]

- \( N \) the number of sensors.
- \( w_n \) the weight applied to the output of the \( n \)th receiver.
- \( a(x_p) \) complex reflectivity of the point target
- \( s(t) \) the emitter signal.
- \( \tau_n \) propagation delay
- \( \tilde{\tau}_n \) focusing delay
Design of Beamformer in presence of Dielectric Wall

The output of the beamformer for a single target is given by:

\[ z_q(t) = \sum_{n=1}^{N} w_n a(x_p) \exp(-\alpha(l_{p1} + l_{np1}))s(t - \tau_n - \tilde{\tau}_n) \]

- \( \alpha \) the attenuation constant of the wall
- \( l_{p1} \) the distance traveled through the wall by the signal \( s(t) \) on transmit.
- \( l_{np1} \) the distance traveled through the wall by the signal \( s(t) \) on the \( n \)th receive.
**General case**

Consider Least-squares error optimization: Mathematically, the optimum weights are given by:

$$\min_w |A^H w - r_d|^2,$$

where

$$A = \begin{bmatrix} d(\theta_1, \omega_1), & d(\theta_2, \omega_2), & \ldots, & d(\theta_P, \omega_P) \end{bmatrix},$$

and

$$r_d = \begin{bmatrix} r_d(\theta_1, \omega_1), & r_d(\theta_2, \omega_2), & \ldots, & r_d(\theta_P, \omega_P) \end{bmatrix}^H.$$

Solution to this problem is the familiar pseudo-inverse:

$$w_{opt} = (AA^H)^{-1} Ar_d.$$
**General case**

In presence of walls: the attenuation term varies w.r.t different direction \( \theta \), noted as:

\[
H = [h(\theta_1, w_1), h(\theta_2, w_2), \ldots, h(\theta_p, w_p)]
\]

Mathematically, the optimum weights are given by:

\[
\min_w |\hat{A}^H w - r_d|^2,
\]

where

\[
\hat{A} = A \odot H
\]

where \( \odot \) represents the Hadamard product.

So the solution to this problem is:

\[
w_{opt} = (\hat{A} \hat{A}^H)^{-1} \hat{A} r_d.
\]
Conclusion

- An overview of beamforming.

- Usefulness of beamforming.

- Various beamforming methods.

- Adaptive vs. data independent beamformers.

- Sample of practical applications.
References


