Size Density Function p.d.f.

$n(R)\,dR = \begin{pmatrix} \text{\# fraction of} \\ \text{particles of size} \\ \text{between } R \text{ \& } R+dR \end{pmatrix}$

$p(R)\,dR = \begin{pmatrix} \text{mass fraction of} \\ \text{particles in the} \\ \text{size range } R \text{ to } R+dR \end{pmatrix}$

$\int_{R_m}^{R_M} n(R)\,dR = 1 \qquad \int_{R_m}^{R_M} p(R)\,dR = 1$

Assuming $\rho_s \neq \rho_s(R)$ i.e. constant density and <u>spherical</u> particles

$$\frac{4}{3}\pi R^3 \rho_s N n(R)\,dR = W\,p(R)\,dR$$

N = total number of particles

W = total mass of particles

$$(1)\quad n(R) = \frac{3W}{4\pi R^3 \rho_s N}\, p(R)$$

↑ usually measured

<u>Mass Average particle size</u> $\overline{R_w}$ is the size of the particle whose mass (ρ_s = const) is the mean of all N particles in the mixture. $m_p \propto R^3$

$$\int_{R_{min}}^{R_{max}} R^3 n(R)\,dR = \overline{R_w}^3$$

$$\frac{3W}{4\pi\rho_s N}\int_{R_{min}}^{R_{max}} p(R)\,dR = \frac{3W}{4\pi\rho_s N} = \overline{R_w}^3$$

$$W = \frac{4}{3}\pi\rho_s \overline{R_w}^3 N$$

Now from (1)

(1a) $$n(R) = \frac{\bar{R}_w^{\,3}}{R^3}\, p(R)$$

$$1 = \int_{R_m}^{R_M} n(R)\,dR = \bar{R}_w^{\,3} \int_{R_m}^{R_M} \frac{p(R)}{R^3}\, dR$$

(2) $$\boxed{\bar{R}_w = \frac{1}{\left[\int_{R_m}^{R_M} \frac{p(R)}{R^3}\, dR\right]^{1/3}}}$$ $$\bar{R}_w = \mu_{-3}^{-1/3}$$

Surface mean size $\bar{R}_s$ is the size of the particle whose surface to volume ratio equals that of all solids in the mixture (This average size is needed in Δp and mass transfer calculations.)

$$S_{ext} = N \int_{R_m}^{R_M} 4\pi R^2\, n(R)\,dR$$

$$V_t = N \int_{R_m}^{R_M} \frac{4}{3}\pi R^3\, n(R)\,dR$$

$$\left(\frac{S_{ex}}{V}\right)_t = \frac{3\int_{R_m}^{R_M} R^2\, n(R)\,dR}{\int_{R_m}^{R_M} R^3\, n(R)\,dR} = \frac{3}{\bar{R}_s}$$

Substitute (1) or (1a)

$$\frac{3\int_{R_m}^{R_M} \frac{p(R)}{R}\, dR}{\int_{R_m}^{R_M} p(R)\,dR} = \frac{3}{\bar{R}_s}$$

$$\boxed{\bar{R}_s = \frac{1}{\int_{R_m}^{R_M} \frac{p(R)}{R}\, dR}}$$ $$\bar{R}_s = \mu_{-1}^{-1}$$

Population Balance

$\psi \, d\Omega$ = fraction of entities in dV with property values in the range $\varphi_i \Leftrightarrow \varphi_i + d\varphi_i$

$$\int_{\Omega_t} \psi \, d\Omega = 1$$ at any given instant fixed x_t

$$\int_{\varphi_{j1}}^{\varphi_{j2}} \psi \, d\varphi_j$$ = ~~number~~ fraction of entities in dV with property value φ_j between φ_{j1} & φ_{j2}

$$B = \frac{\text{birth of entities}}{(\text{unit time})(\text{unit volume})(\text{unit property change})} = \frac{\text{birth}}{\Delta}$$

$$D = \frac{\text{death}}{\Delta}$$

Microscopic (Differential Element)

Accumulation = net generation

$$\frac{d}{dt}\int_{\Omega} \psi \, d\Omega = \int_{\Omega} (B - D) \, d\Omega$$

By Leibnitz rule $\quad u_i = \frac{dx_i}{dt} \quad v_i = \frac{d\varphi_i}{dt}$

$$\int_{\Omega} \left(\frac{\partial \psi}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(u_i \psi) + \sum_{i=1}^{N} \frac{\partial}{\partial \varphi_i}(v_i \psi) + D - B \right) d\Omega = 0$$

Since Ω is chosen arbitrary

$$\frac{\partial \psi}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(u_i \psi) + \sum_{i=1}^{N} \frac{\partial}{\partial \varphi_i}(v_i \psi) + D - B = 0$$

$$\frac{D\psi}{Dt} + (\vec{u}\,\nabla)\psi$$

$$\frac{\partial\psi}{\partial t} + \nabla\cdot(\vec{u}\,\psi) + \sum_{i=1}^{N} \frac{\partial}{\partial\varphi_i}(v_i\,\psi) + D - B = 0$$

$$\bar{\psi} = \frac{1}{V}\int_V \psi\, dV$$

Recall Reynolds Transport Theorem

$$\frac{d}{dt}\int_V \psi\, dV = \int_V \left[\frac{\partial\psi}{\partial t} + \nabla\cdot\vec{u}\,\psi\right] dV =$$

$$= \int_V \frac{\partial\psi}{\partial t}\, dV + \oint_S \psi\, \vec{u}\cdot\vec{n}\, ds =$$

$$= \int_V \frac{\partial\psi}{\partial t}\, dV + \oint_S \psi\, \frac{\partial R}{\partial n}\, ds$$

$$\int_V \frac{\partial\psi}{\partial t}\, dV + \int_V \nabla(\vec{u}\,\psi)\, dV + \int_V \sum_{i=1}^{N} \frac{\partial}{\partial\varphi_i}(v_i\,\psi)\, dV + \int_V (D - B)\, dV = 0$$

$$\int_V \frac{\partial\psi}{\partial t}\, dV + \oint_S \psi\, \vec{v}\cdot\vec{n}\, dS + \int_V \sum_{i=1}^{N} \frac{\partial}{\partial\varphi_i}(v_i\,\psi)\, dV + \int_V (D-B)\, dV = 0$$

V – fixed in space & time

$$\frac{\partial}{\partial t}\int_V \psi\, dV + \psi_{out}\, Q_{out} - \psi_{in}\, Q_{in} + \sum_{i=1}^{N} \frac{\partial}{\partial\varphi_i}\left(v_i \int_V \psi\, dV\right) + \int_V (D-B)\, dV = 0$$

$$\frac{\partial}{\partial t}\left(V\,\bar{\psi}\right) + \psi_{out}\, Q_{out} - \psi_{in}\, Q_{in} + V\sum_{i=1}^{N} \frac{\partial}{\partial\varphi_i}\left(v_i\,\bar{\psi}\right) + \bar{D} - \bar{B} = 0$$

At steady state: $V = \text{const}$ for a

$$V\sum_{i=1}^{N} \frac{\partial}{\partial\varphi_i}\left(v_i\,\bar{\psi}\right) + \bar{D} - \bar{B} = \psi_{in}\, Q_{in} - \psi_{out}\, Q_{out}$$

Recall general population balance at steady state

$$V \sum \frac{d}{dy_i}\left(v_i \bar{\psi}\right) + \bar{D} - \bar{B} = \psi_{in} Q_{in} - \psi_{out} Q_{out}$$

CSTR $\quad \bar{\psi} = \psi_{out}$

$y_i = R$

$$\boxed{V \frac{d}{dR}\left[n \frac{dR}{dt}\right] = \dot{V}_{in}\, n_{in} - \dot{V}_{out}\, n_{out}}$$

Population Balance for Fluidized Beds

Interval dR (Steady state)

$$\begin{pmatrix}\text{number of particles}\\ \text{of size between } R \,\&\, R+dR\\ \text{entering the bed per unit time}\end{pmatrix} - \begin{pmatrix}\text{number of particles}\\ \text{of size between } R \,\&\, R+dR\\ \text{leaving the bed per}\\ \text{unit time}\end{pmatrix}$$

$$+ \begin{pmatrix}\text{number of particles}\\ \text{of size between } R \,\&\, R+dR\\ \text{generated in the bed}\\ \text{per unit time}\end{pmatrix} - \begin{pmatrix}\text{number of particles}\\ \text{of size between } R \,\&\, R+dR\\ \text{destroyed per unit}\\ \text{time}\end{pmatrix}$$

$$+ \begin{pmatrix}\text{number of particles}\\ \text{of size } R-dR \text{ growing}\\ \text{into size } R \text{ to } R+dR\\ \text{per unit time}\end{pmatrix} - \begin{pmatrix}\text{number of particles}\\ \text{of size } R \,\&\, R+dR\\ \text{outgrowing this size range}\\ \text{per unit time}\end{pmatrix} = 0$$

~~steady state $\dot W_{in} = \dot W_{out}$~~

$$\dot N_{in}(R) - \dot N_{out}(R) + \dot B(R)^{0} - \dot D(R)^{0} + \frac{N(R)\,\mathcal{R}}{dR} - \frac{N(R+\Delta R)\,\mathcal{R}}{dR} = 0$$

$$\dot N_{in}\, n_{in}(R)\,dR - \dot N_{out}\, n_{out}(R)\,dR + N n(R)\,dR\,\frac{\mathcal{R}}{dR} - N n(R+dR)\,\frac{\mathcal{R}_{R+dR}}{dR} = 0$$

CSTR assumption $n_{out}(R) = n(R)$

$$\dot N_{in}\, n_{in}(R) - \dot N_{out}\, n(R) - N\frac{d}{dR}\left[n\,\mathcal{R}\right] = 0$$

$N n(R)\, dR = N(R)$ — number of particles of size $R - R+dR$

$W p(R)\, dR = W(R)$ — weight of particles of size R to $R+dR$

~~$N = \frac{W}{\frac{4}{3}\pi R^3 \rho}$~~

$W(R) = \frac{4}{3}\pi R^3 \rho N(R)$

$W = \frac{4}{3}\pi \bar{R}^3 \rho N$

$$\dot{N}_{in}\, n_{in}(R)\,\Delta R - \dot{N}_{out}\, n_{out}(R)\,\Delta R + N\, n(R)\, \mathcal{R}\Big|_R$$

$$- N\, n(R)\, \mathcal{R}\Big|_{R+\Delta R} = 0$$

$$\dot{N}_{in}\, n_{in}(R) - \dot{N}_{out}\, n_{out}(R) - N \frac{d}{dR}\left[n(R)\, \mathcal{R} \right] = 0$$

$$n(R) = \frac{3\, W\, p(R)}{N\; 4\pi R^3 \rho_s} \qquad n_{in} = \frac{3\, F_{in}\, p_{in}(R)}{\dot{N}_{in}\; 4\pi R^3 \rho_s}$$

$$\cancel{\dot{N}_{in}} \frac{\cancel{3} F_{in}\, p_{in}(R)}{\cancel{\dot{N}_{in}}\; 4\pi R^3 \rho_s} - \cancel{\dot{N}_{out}} \frac{\cancel{3}\, F_{out}\, p_{out}(R)}{\cancel{\dot{N}_{out}}\; 4\pi R^3 \rho_s} - N \frac{\cancel{3} W}{N 4\pi \rho_s} \frac{d}{dR}\left[\frac{p(R) \mathcal{R}}{R^3} \right] = 0$$

$$p_{out} = p$$

$$F_{in}\, p_{in}(R) - F_{out}\, p(R) - R^3 W \frac{d}{dR}\left[\frac{p(R)\, \mathcal{R}}{R^3} \right] = 0$$

$$F_{in}\, p_{in}(R) - F_{out}\, p(R) - W \frac{d}{dR}\left[p(R)\, \mathcal{R} \right] + \frac{3W}{R} p(R) \mathcal{R} = 0$$

$$\left(\begin{array}{c} \text{mass ~~fraction~~} \\ \text{in the size interval} \end{array} \right) = \rho_s \left(\begin{array}{c} \text{number of particles} \\ \text{in the size interval} \end{array} \right) \left(\begin{array}{c} \text{volume increase of} \\ \text{a particle in } \Delta t \end{array} \right)$$

$$= \rho_s \frac{3\, W\, p(R) \cancel{dR}}{4\pi \rho_s\, R^3}\; 4\pi R^2 \frac{dR}{dt}$$

$$= \frac{3\, W\, p\, \mathcal{R}}{R}$$

$$N(R) = \frac{W(R)}{\frac{4}{3}\pi R^3 \rho_s} = \frac{3\, W(R)}{4\pi R^3 \rho_s} = \frac{3 W\, p(R)\, dR}{4\pi R^3 \rho_s}$$

$$n(R)\,dR = \frac{N(R)}{N} = \frac{3\, W\, p(R)\, dR}{N\, 4\pi R^3 \rho_s}$$

For uniform solid density

$$N = \frac{W}{\frac{4}{3}\pi \bar{R}^3 \rho_s}$$

$$n(R) = \frac{3 W\, p(R)}{N\, 4\pi R^3 \rho_s}$$

Similarly ($\rho_s = \text{const}$)

$$\dot{n}_{in}(R) = \frac{3\, F_{in}\, p_{in}(R)}{\dot{N}_{in}\, 4\pi R^3 \rho_s}$$

$$\dot{N}_{in} \frac{3 F_{in}\, p_{in}(R)}{\dot{N}_{in}\, 4\pi R^3 \rho_s} - \dot{N}_{out} \frac{3 F_{out}\, p(R)}{\dot{N}_{out}\, 4\pi R^3 \rho_s} - N \frac{d}{dR}\left[\frac{3 W p(R)}{N 4\pi R^3 \rho_s} R\right] = 0$$

$$\frac{3}{4\pi R^3 \rho_s}\left[F_{in}\, p_{in}(R) - F_{out}\, p(R)\right] - \frac{3 W}{4\pi R^3 \rho_s} \frac{d}{dR}\left[p R\right]$$

$$+ \frac{9 W p R}{4 \pi R^4 \rho_s} = 0$$

$$\boxed{F_{in}\, p_{in} - F_{out}\, p - W \frac{d}{dR}(p R) + 3 \frac{W}{R} p R = 0}$$

basic balance in terms of weight (mass)