\[ \frac{dP}{dt} = -kP \]

\[ m(t) = \frac{P(t)}{A} \]

\[ \frac{dA}{dt} = \text{mass flow rate into the system} \]

\[ \text{fluid} \]

\[ \frac{dV}{dt} = \text{mass flow rate out of the system} \]

\[ \text{fluid} \]

\[ \text{mass} \]

\[ \text{flow rate} \]

\[ \text{system} \]

\[ \text{mass} \]

\[ \text{flow rate} \]

\[ \text{system} \]

\[ \text{mass} \]

\[ \text{flow rate} \]

\[ \text{system} \]

\[ \text{mass} \]

\[ \text{flow rate} \]

\[ \text{system} \]

\[ \text{mass} \]

\[ \text{flow rate} \]

\[ \text{system} \]
\[
\begin{align*}
K &= \frac{\beta}{A} \\
\frac{dx}{dt} &= -Kx \quad \text{(21)} \\
R(t) &= \frac{K}{\lambda} e^{-\lambda t} \\
\frac{dR}{dt} &= -\lambda R(t) \quad \text{(22)}
\end{align*}
\]

For $\lambda > 0$, the solution to the above differential equation is:

\[
R(t) = \frac{K}{\lambda} e^{-\lambda t}
\]

The exponential term $e^{-\lambda t}$ ensures that $R(t)$ approaches zero as $t$ increases. This is a fundamental property of the exponential decay process.
For a single one feed of radius $R_0$ one gets

$$\frac{W}{\dot{W}_0} = \int_{R_0}^{R(t=\infty)} \frac{R^3}{R(R)R_0^3} I(R, R_0) \, dR$$  \hspace{1cm} (A1) (4)

where

$$I(R, R_0) = \exp \left[ - \int_{R_0}^{R} \frac{\dot{W}_0 / W + x(R)}{R(R)} \, dR \right]$$  \hspace{1cm} (A2)

The net distribution in the bed is

$$P_1(R) = \frac{\dot{W}_0}{W |R(R)|} \frac{R^3}{R_0^3} I(R, R_0)$$  \hspace{1cm} (A3) (2)

Overall balance gives

$$\dot{W}_1 + \dot{W}_2 - \dot{W}_0 = \int \frac{3W P_1(R) R(R) \, dR}{R}$$  \hspace{1cm} (A4)

$$P_2(R) = \frac{W}{\dot{W}_2} x(R) P_1(R)$$  \hspace{1cm} (A5)

**Example 1:** We want to produce 150 kg/h of Si. Si is a fluidized bed.

$$SiH_4 = Si + 2H_2$$

Seed particles at $R = 100 \mu m$ are fed into the reactor. We want to produce a mixture mean particle size of $\bar{R}_s = 1000 \mu m$. The growth rate is $100 \mu m/h$.

Find the size of the reactor and seed feed rate $\dot{W}_0$. 

$$W_1$$, and seed feed rate $\dot{W}_0$.
First recall that

\[ m_R(R) \, dR = \text{mass fraction of solids in the bed between } R \text{ and } R + dR \]

The mass fraction of solids in the bed between two radii \( R \) and \( R + dR \) is:

\[ m_R(R) \, dR = \frac{W \, m_R(R) \, dR}{N \frac{4}{3} \pi R^3} \]

Mean mass hole radius based on external surface

is defined by:

\[ S_\text{ox} = N \frac{4}{3} \pi R_s^2 = N \int_{R_m}^{R_m} R^2 m(R) \, dR = \frac{3W}{R_s} \int_{R_m}^{R_m} \frac{m_R(R) \, dR}{R} \]

\[ V = N \frac{4}{3} \pi R_s^3 = N \int_{R_m}^{R_m} \frac{4}{3} \pi R^3 m(R) \, dR = \frac{W}{R_s} \int_{R_m}^{R_m} \frac{m_R(R) \, dR}{R} = \frac{W}{R_s} \]

\[ \frac{S_{\text{ox}}}{V} = \frac{3}{R_s} = \frac{3W}{R_s} \int_{R_m}^{R_m} m_R(R) \, dR \]

- Size of pan hole whose surface to volume ratio equals that of all solids in the mixture.

Other averages can be defined

\[ \bar{R}_W = \frac{1}{\int_{R_m}^{R_m} \frac{m_R(R) \, dR}{R^3}} = \text{weight average pan hole radius is the one of pan hole whose weight is the mean of all } W \text{ pan holes in the mixture} \]
Now let us solve the problem:

From (A2)

\[ I(R, R_0) = e^{\frac{\dot{w}_1}{kW}(R_0 - R)} \]

Since \( R(R) = k = 100 \mu m/h \)

From (A3)

\[ P(R) = \frac{\dot{w}_2}{WK} \frac{R^3}{R_0^3} e^{\frac{\dot{w}_1}{kW}(R_0 - R)} \]

for \( R > R_0 \)

From (A4)

\[ \frac{\dot{w}_1}{\dot{w}_0} = 1 + \int R_0^R \frac{3}{R_0^3} e^{\frac{\dot{w}_1}{kW}R_0} R^2 e^{-\frac{\dot{w}_1}{kW}R} dR \]

\[ \frac{\dot{w}_1}{\dot{w}_0} = 1 + 3 \frac{WK}{\dot{w}_1 R_0} + 6 \left( \frac{WK}{\dot{w}_1 R_0} \right)^2 + 6 \left( \frac{WK}{\dot{w}_1 R_0} \right)^3 \]

\[ \bar{R}_S = \frac{1}{\frac{\dot{w}_0}{WK R_0^3} e^{\frac{\dot{w}_1}{kW}R_0} \int R_0^R R^2 e^{-\frac{\dot{w}_1}{kW}R} dR} \]

\[ = \frac{1}{\frac{\dot{w}_0}{WK R_0^3} e^{\frac{\dot{w}_1}{kW}R_0} \left[ -R^2 \frac{kW}{\dot{w}_1} e^{-\frac{\dot{w}_1}{kW}R} - 2R \left( \frac{kW}{\dot{w}_1} \right)^2 e^{-\frac{\dot{w}_1}{kW}R} - 2 \left( \frac{kW}{\dot{w}_1} \right)^3 e^{-\frac{\dot{w}_1}{kW}R} \right]} \]

\[ = \frac{WK}{\dot{w}_0} \left[ \frac{kW}{\dot{w}_1 R_0} + 2 \left( \frac{kW}{\dot{w}_1 R_0} \right)^2 + 2 \left( \frac{kW}{\dot{w}_1 R_0} \right)^3 \right] \]

\[ = \frac{3WK}{\dot{w}_0 \left( \frac{\dot{w}_1}{\dot{w}_0} - 1 \right)} \]

\[ \bar{R}_S = \frac{3 \frac{WK}{\dot{w}_0}}{\frac{\dot{w}_1}{\dot{w}_0} - 1} \]
\( \frac{R_s}{R_0} = \frac{3 \frac{W_K}{w_0 R_0}}{\frac{w_t}{w_0} - 1} \)

\[
\frac{1000}{100} = \frac{3 \frac{W \times 100}{w_0 \times 100}}{\frac{w_t}{w_0} - 1} = \frac{3 \frac{W}{w_0}}{\frac{w_t}{w_0} - 1}
\]

\[10 = \frac{3 \frac{W}{w_t - w_0}}{100} = \frac{3 \frac{W}{w_t - w_0}}{100} \]

\[W = \frac{10}{3} (w_t - w_0)\]

Product rate in kg

\[150 = w_t - w_0\]

Required mass of bed

\[W = \frac{10}{3} \times 150 = 500 \text{ kg}\]

\[w_0 = \frac{150}{w_t - 150}\]

\[\frac{W_K}{w_t R_0} = \frac{500 \times 100}{w_t \times 100} = \frac{500}{w_t}\]

\[\frac{w_t}{w_t - 150} = 1 + 3 \left( \frac{500}{w_t} \right) + 6 \left( \frac{500}{w_t} \right)^2 + 6 \left( \frac{500}{w_t} \right)^3\]

Solve by trial and error

\[w_t = 150.5 \text{ kg/h} \quad \text{exit stream}\]

\[w_0 = 0.5 \text{ kg/h} \quad \text{feed stream or needs}\]

\[W = 500 \text{ kg} \quad \text{bed size}\]

Note: If we wanted to grow particles up to \( R_t = 200 \mu_m \), we would need a bed of \( W = 100 \text{ kg} \). The exit stream would be \( w_t = \frac{184.5}{w_t} \text{ kg/h} \), and needs feed rate \( w_0 = 34.5 \text{ kg/h} \).
Example 2: We are producing TCS in a
fluidized bed

\[ \text{Si}(s) + \text{K}_2 \text{O}(g) + 3 \text{Si} \text{Cl}_4(g) = 4 \text{Si} \text{HCl}_3(g) \]

The solids feed consists of a single particle
size \( R_0 = 200 \mu\text{m} \). These particles under
reactor conditions would react completely in 8 h.
However, the particles shrink to 50 \( \mu\text{m} \)
and are then entrained out of the bed.

How large a bed do we need to convert
120 kg/h of Si to TCS?

Assume linear shrinkage

\[ R(t) = R_0 e^{-kt} \]

\[ \frac{R_0 - R}{R_0} = t \]

\[ k = \frac{R_0 - R}{t} = \frac{200 - 50}{8} = 25 \mu\text{m/h} \]

This reaction is equivalent to plug flow of solids
since all solids stay the same time in the
reactor i.e.

\[ t = \frac{R_0 - R}{k} = \frac{200 - 50}{25} = 6 \text{ (h)} \]

Last time (Lecture 9) we developed the
equation for plug flow of solids and the
constant shrinkage rate

\[ \frac{W}{K} \frac{1}{\omega R_0} = \frac{1}{4} \left[ 1 - \left( \frac{R}{R_0} \right)^4 \right] = \frac{1}{4} \left[ 1 - \left( 1 - \frac{kt}{R_0} \right)^4 \right] \]

\[ \frac{25 W}{200 \omega_0} = \frac{1}{4} \left[ 1 - \left( \frac{50}{200} \right)^4 \right] \]

\[ W \quad = \quad 2 \left[ 1 - \frac{1}{4^2} \right] \]
At the same time since the number of particles in the feed and eluted stream is the same
\[
\frac{\dot{w}_2}{\dot{w}_0} = \left(\frac{R}{R_0}\right)^3 = \left(\frac{50}{200}\right)^3 = \frac{1}{4^3}
\]
we want to use 120 kg/l of Si LE
\[
\dot{w}_0 - \dot{w}_2 = 120
\]
\[
\dot{w}_0 \left[1 - \frac{\dot{w}_2}{\dot{w}_0}\right] = 120 \quad \dot{w}_0 = \frac{120}{1 - \frac{1}{4^3}} = 121.9 \approx 122 \text{ kg}
\]
\[
W = 2 \dot{w}_0 \left[1 - \frac{1}{4^3}\right] = 2 \times 120 \frac{1 - \frac{1}{4^3}}{1 - \frac{1}{4^3}} = 242.9 = 243 \text{ kg}
\]

Using the generalized equations presented here
\[
I(R, R_0) = H(R - R_{min})
\]
\[
H(u) = \begin{cases} 0 & u < 0 \\ 1 & u > 0 \\ \text{undefined} & u = R = R_0 \end{cases}
\]
\[
M(R) = \frac{\dot{w}_0}{W} \left(\frac{R}{R_0}\right)^3 \frac{1}{K} \quad H(R - R_{min})
\]
\[
\dot{w}_0 - \dot{w}_2 = 120 \text{ kg/l}
\]

Eq. (A.4) gives
\[
\frac{\dot{w}_2}{\dot{w}_0} = \left(\frac{R}{R_0}\right)^3
\]

Eq. (A.1) gives
\[
\frac{W}{\dot{w}_0 R_0} = \frac{1}{4} \left[1 - \left(\frac{R}{R_0}\right)^4\right]
\]

These are the same as plug flow equations.
Extensions to multi-stage feed, ebullation, etc. can readily be obtained

\[ I(R; R_0) = \exp \left[ -\int_{R_0}^{R} \frac{\dot{W}_i}{\dot{W}} + \chi(R) \, dR \right] \]

Let \( R_m \) be the smallest size in the feed, \( R_f \) the largest size in the feed.

\( W_0(R) \) = fraction (mol.) of particles of size \( R \) in feed

For growing particles:

\[ \frac{W}{\dot{W}_0} = \int_{R_m}^{R_{i=\infty}} \frac{R^3}{|R(R)|} I(R; R_m) \, dR \int_{R_m}^{R} \frac{W_0(R_i) \, dR_i}{R_i^3 I(R_i; R_m)} \]

For shrinking particles:

\[ \frac{W}{\dot{W}_0} = \int_{R(t=0)}^{R_f} \frac{R^3}{|R(R)|} I(R; R_m) \, dR \int_{R_f}^{R} \frac{W_0(R_i) \, dR_i}{R_i^3 I(R_i; R_m)} \]

For growing particles:

\[ N_i(R) = \frac{\dot{W}_0 \, R^3}{W |R(R)|} I(R; R_m) \int_{R_m}^{R} \frac{W_0(R_i) \, dR_i}{R_i^3 I(R_i; R_m)} \]

For shrinking particles:

\[ N_s(R) = \frac{\dot{W}_0 \, R^3}{W |R(R)|} I(R; R_m) \int_{R_f}^{R} \frac{W_0(R_i) \, dR_i}{R_i^3 I(R_i; R_m)} \]

and

\[ \dot{W}_1 + \dot{W}_2 - \dot{W}_0 = \int_{R} ^{R_f} 3 \frac{\dot{W}_0 R^3}{W |R(R)|} \frac{2(R) \, dR}{dR} \]


Objective: Study the following:

1. **Molecular Flow**
   - **Initial Assumption**:
   - \[ \frac{W}{F_0} = \int_0^{x_0} \frac{dR}{(-R_0)} \]
   - \[ \frac{dR}{dt} = -k \]

2. **Energy Consideration**:
   - \[ -R_0 = \frac{4 \pi R^2 P_o (-dR)}{\frac{4}{3} \pi R^3 P_o} = \frac{3}{2} \frac{R}{R_0} = \frac{3}{2} \left( \frac{dR}{dt} \right) \]

3. **Solution**:
   - \[ x_0 = \frac{\frac{4}{3} \pi R_0^3 - \frac{4}{3} \pi R^3}{\frac{4}{3} \pi R_0^3} = 1 - \left( \frac{R}{R_0} \right)^3 \]

4. **Work Calculation**:
   - \[ dW = -\frac{3R^2}{R_0^3} dR \]
   - \[ \frac{W}{F_0} = \int_{R_0}^{R} \frac{R^3}{R_0^3 \left( \frac{dR}{dt} \right)} dR \]

5. **Integration**:
   - \[ \frac{W}{F_0} = -\int_{R_0}^{R} \frac{3R^2 \cdot R}{R_0^3 \cdot 3k} dR = \frac{1}{4k R_0^3} \int_{R_0}^{R} R^3 dR \]
   - \[ = \frac{1}{4k R_0^3 k} \left[ R_0^4 - R^4 \right] = \frac{R_0}{4k} \left[ 1 - \left( \frac{R}{R_0} \right)^4 \right] \]

6. **Conditions**:
   - \[ \frac{dR}{dt} = -k \]
   - \[ k = R_0 - k_0 \]
   - \[ t = 0 \quad R = R_0 \quad \frac{R}{R_0} = 1 - \frac{k}{k_0} t \]

7. **Final Solution**:
   - \[ \frac{W}{F_0} = \frac{R_0}{4k} \left[ 1 - (1 - \frac{k}{k_0})^4 \right] \]