5.2 Mixing in a Pipeline

When instead of a reactor problem we deal with problems of mixing of one material that flows after another as both are being pumped through the same pipe line, we use the dispersion or Taylor diffusion model (depending whether the flow is turbulent or laminar) to describe the spreading of the material in the axial direction.

The governing equations in the coordinate system that moves at the mean velocity of flow can be written according to the diffusion equation:

$$\frac{\partial \tilde{c}}{\partial \theta'} = \frac{1}{Pe} \frac{\partial^2 \tilde{c}}{\partial \zeta^2} \tag{122}$$

where $Pe = \frac{\bar{u} L}{D_{app}}$

and D_{app} is either the dispersion coefficient, E_z, or Taylor diffusivity.

If we try to describe an impulse response in a very long pipe, then all the material was concentrated originally at the plane $\zeta = 0$. At the same time

$$\theta' = 0; \quad \tilde{c} = 0 \text{ except at } \zeta = 0 \tag{123a}$$

$$\theta' > 0; \quad \zeta \to \infty \quad \tilde{c} \to 0 \tag{123b}$$

no material can reach axial position of infinity at a finite time which is expressed by the second condition above. Finally, since the mass m_t of the tracer material injected at time zero at the plane at zero axial position must be conserved we have the last condition:

$$\int_{-\infty}^{\infty} \tilde{c}\, d\zeta = \frac{m_t}{A\, C_o\, L} \tag{124}$$

where $A = \pi R^2$ is the cross-sectional area of the system, L is the length with respect to which we have dimensionalized the axial coordinate, C_o is a normalizing concentration.

The solution to the above problem can be obtained by either

a) taking the Laplace transform of the PDE, and B.C., solving the resulting ODE and inverting the solution, or

b) by similarity transform i.e. by introducing new variables

$$\eta = \frac{\zeta}{\sqrt{\theta'}} \; and \; u = \tilde{c}\sqrt{\theta'}$$

The solution is

$$\tilde{c}(\zeta, \theta') = \frac{m_t}{\pi R^2 L C_o} \frac{1}{2\sqrt{\frac{\pi \theta'}{Pe}}} e^{-\frac{\zeta^2}{4\frac{\theta'}{Pe}}} \tag{125}$$

If we consider $\tilde{c}\, C_o = C$ actual concentration and turn back to the fixed coordinate system

$$\xi = \frac{z}{L} = \zeta + \theta \quad ; \quad \theta = \theta' = \frac{t}{\bar{t}} = \frac{\bar{u}\, t}{L} \tag{126}$$

$$C(\xi, \theta) = \frac{m_t}{\pi R^2 L} \sqrt{\frac{Pe}{4\, \pi \theta}}\; e^{-\frac{Pe(\xi - \theta)^2}{4\, \theta}} \tag{127}$$

$$C(z, t) = \frac{m_t}{\pi R^2 L} \sqrt{\frac{LPe}{4\pi \bar{u}\, t}}\; e^{-\frac{Pe(z - \bar{u} t)^2}{4 \bar{u} L t}} \tag{128}$$

or taking $Pe = \frac{\bar{u}\, L}{D_{app}}$

$$C(z, t) = \frac{m_t}{\pi R^2} \sqrt{\frac{1}{4\pi D_{app} t}}\; e^{-\frac{(z - \bar{u}\, t)^2}{4 D_{app} t}} \tag{128a}$$

The impulse response at $z = L$, $G(t)$ (not necessarily the age density function because the system may be open) is now given by

$$G(t) = \frac{\pi R^2 \bar{u}\, C}{m_t} = \sqrt{\frac{\bar{u}^2}{4\, \pi D_{app} t}}\; e^{-\frac{(L - \bar{u} t)^2}{4\, D_{app} t}} \tag{129}$$

and

$$G_\theta(\theta) = \bar{t}\, G(t) = \sqrt{\frac{L^2}{4\, \pi D_{app} \frac{L}{\bar{u}}\, \theta}}\; e^{-\frac{L^2 (1 - \theta)^2}{4 D_{app} \frac{L}{\bar{u}} \theta}} \tag{130a}$$

$$G_\theta(\theta) = \sqrt{\frac{Pe}{4\pi\theta}}\, e^{-\frac{Pe(1-\theta)^2}{4\theta}} \tag{130}$$

$$Pe = \frac{\bar{u}\,L}{D_{app}} \propto L$$

Those that had probability theory will recall that the Gaussian density function is given by:

$$f(\theta) = \frac{1}{\sigma_D\sqrt{2\pi}}\, e^{-\frac{(\theta-\mu)^2}{2\sigma_D^2}} \tag{131}$$

It can readily be shown that in long beds $G_\theta(\theta)$ given by eq (130) is small for all θ except those in the vicinity of $\theta = 1$. In the first approximation one can represent the $G_\theta(\theta)$ by a Gaussian density function

$$G_\theta(\theta) \approx \frac{1}{\sigma_D\sqrt{2\pi}}\, e^{-\frac{(\theta-1)^2}{2\sigma_D^2}} \tag{132}$$

where

$$\sigma_D^2 = \frac{2}{Pe} = \frac{2\,D_{app}}{\bar{u}\,L} \tag{133}$$

This shows that given the flow velocity profile u, then the mean velocity $\bar{u}$ and the apparent axial dispersion coefficient $\frac{\bar{u}^2 R^2}{48\,D}$ is fixed, if Taylor diffusion model is applicable.

The relative spread of the curve around the mean then is reduced as the length L between the injection and monitoring station is increased.

The absolute spread, however,

$$\sigma^2 = \sigma_D^2\,\bar{t}^2 = \frac{2}{Pe}\frac{L^2}{\bar{u}^2} = \frac{2\,D_{app}L}{\bar{u}^3} \tag{134}$$

increases as the length of the conduit is increased. If the dispersion model holds, the increase of the spread measured by $\sigma = \sqrt{\sigma^2}$ is proportional to $L^{1./2}$.

Step response is now given by:

$$F_s(\theta) = \frac{1}{\sigma_D\sqrt{2\pi}} \int_{-\infty}^{\theta} e^{-\frac{(\theta-1)^2}{2\sigma_D^2}}\, d\theta \tag{135}$$

$$x = \frac{\theta - 1}{\sqrt{2}\ \sigma_D}, \quad d\theta = \sqrt{2}\,\sigma_D\, dx, \quad \theta = 1 + \sqrt{2}\,\sigma_D\ x$$

$$F_s(\theta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{\theta-1}{\sqrt{2}\sigma_D}} e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^{o} e^{-x^2} dx + \int_{o}^{\frac{\theta-1}{\sqrt{2}\sigma_D}} e^{-x^2} dx \right]$$

$$= \frac{1}{2} \left[\frac{2}{\sqrt{\pi}} \int_{o}^{\infty} e^{-x^2} dx + \frac{2}{\sqrt{\pi}} \int_{o}^{\frac{\theta-1}{\sqrt{2}\sigma_D}} e^{-x^2} dx \right]$$

$$= \frac{1}{2}\left[1 + erf\left(\frac{\theta - 1}{\sqrt{2}\ \sigma_D}\right)\right] = \frac{1}{2}\left[1 - erf\left(\frac{1-\theta}{\sqrt{2}\ \sigma_D}\right)\right]$$

$$= \frac{1}{2}\, erfc\left(\frac{1-\theta}{\sqrt{2}\ \sigma_D}\right) \text{ with } \ \sigma_D = \sqrt{\frac{2}{Pe}}$$

which is the normal distribution.

$$F_s(\theta) = \begin{matrix} \frac{1}{2}\left[1 - erf\left(\frac{1-\theta}{\sqrt{2}\sigma_D}\right)\right] for\, \theta < 1 \\ \frac{1}{2}\left[1 + erf\left(\frac{\theta-1}{\sqrt{2}\sigma_D}\right)\right] for\, \theta > 1 \end{matrix} \qquad (136)$$

This formula can be used to

a) determine after switching at $\theta = 0$ at position z =0 from fluid A to fluid B what fraction of fluid B appears at the outflow and how long a time it takes until the outflow contains 95% or more of fluid B.

b) determine the length over which one has a mixture between p % of fluid A and p % of fluid B.

c) other problems of similar type.