CSE547T Class 5

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1 Expressiveness, Part Deux

NFAs are nifty. But there is still one thing NFAs are not very good at: composition!

- DFA composition (at least the intersection) involves a big blowup in state count; union can be done the same way but adds just as much complexity.
- NFA composition ought to be a lot easier (in some cases).
- Suppose we have NFAs M_1 and M_2, and we want to accept $L(M_1) \cup L(M_2)$.
- Intuitively, we should be able to say: "Given input x, nondeterministically choose M_1 or M_2, then feed x to the chosen machine."
- Unfortunately, we can't quite say that yet. Example: let's try to wire together the following three NFAs into a union machine...

- Note that alleged union accepts aaa, while none of the components do. (Exercise: fix it!)
- Once again, we are limited in our expressive power.

How do we fix this problem?

2 Null Transitions

We need an extension that lets us hook together NFAs without having to merge their transitions.

- Intuitively, we want to add "free transitions" from one state to another.
- "Free" means not using up any input characters.
- We will label these free transitions with ε in state diagrams, so that it's clear that they don't consume input.
- A fix for our example above:

- We call this new feature "ε-transitions" or "null transitions."
- Here's another nifty trick you can do with null transitions: given machines M_1 and M_2, you can trivially concatenate them. Example:

OK, null transitions are intuitively what we want. But do they make formal sense?

3 Formalizing Null Transitions

We'll now give a precise definition of null transitions.

- An ε-NFA is a 5-tuple $M = (Q, \Sigma, q_0, A, \delta)$, where
- Q is a set of states,
- Σ is the input alphabet,
- q_0 is a starting state,
- A is a set of accepting states,
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \to 2^Q$ is a transition function.
- For $a \in \Sigma$, $\delta(q, a)$ is defined as for an ordinary NFA.
- $\delta(q, \varepsilon)$ is the set of states reachable from q by taking one null transition.

- Extension of δ to state sets is again natural. For $a \in \Sigma$,

$$\delta(\psi, a) = \bigcup_{r \in \psi} \delta(r, a),$$

and

$$\delta(\psi, \varepsilon) = \bigcup_{r \in \psi} \delta(r, \varepsilon).$$

To simplify notation, we will need one more definition:

- Let ψ be a subset of Q for a ε-NFA. The *ε-closure of ψ*, denoted $\varepsilon(\psi)$, is the set of all states reachable from ψ by *zero or more* null transitions.
- A couple of facts worth noting (trivial proofs from defn of ε-closure):
  1. $\varepsilon(\varepsilon(\psi)) = \varepsilon(\psi)$.
  2. $\varepsilon(\psi) = \bigcup_{r \in \psi} \varepsilon(\{r\})$.

Onwards to δ^*!

- For any $q \in Q$, $\delta^*(q, \varepsilon) = \varepsilon(\{q\})$.
- For any $q \in Q$ and string $x = ya$,

$$\delta^*(q, x) = \varepsilon(\delta(\delta^*(q, y), a)).$$

- Once again, we want to extend δ^* to sets of states. We use the following "obvious" definition:

$$\delta^*(\psi, x) = \bigcup_{r \in \psi} \delta^*(r, x).$$

- Again, there is something to prove! Can show that

$$\delta^*(\psi, \varepsilon) = \varepsilon(\psi)$$

and that for $x = ya$,

$$\delta^*(\psi, x) = \varepsilon(\delta(\delta^*(\psi, y), a)).$$

4 Equivalence of ε-NFAs and Ordinary NFAs

I claim that adding null transitions increases expressiveness but has no effect on power.

- I will leave as an exercise checking that every NFA has an equivalent ε-NFA – the definitions reduce to those for an NFA when $\varepsilon(\{q\}) = \{q\}$ for every $q \in Q$.
- **Claim**: for every ε-NFA M, there exists an NFA N with $L(N) = L(M)$.
- Let ε-NFA $M = (Q, \Sigma, q_0, A_M, \delta_M)$ be given.
- Define an NFA $N = (Q, \Sigma, q_0, A_N, \delta_N)$ as follows.

- Q, Σ, and q_0 are all the same for M and N.
- For reasons that will become clear in a bit, we define A_N as follows:
  - If $\varepsilon(q_0)$ contains an accepting state of M, $A_N = A_M \cup \{q_0\}$.
  - Otherwise, $A_N = A_M$.
- Finally, we define
$$\delta_N(q, a) = \varepsilon(\delta_M(\varepsilon(\{q\}), a)).$$
- Using our set notation, it follows that
$$\delta_N(\psi, a) = \varepsilon(\delta_M(\varepsilon(\psi), a)).$$

Intuitively, what's going on? To read a in state q, the ε-NFA can take any available null transitions, then read a, then take any *new* available null transitions. The NFA therefore jumps directly to all possible states reachable from q on a.
Example:

5 Proof of Equivalence

OK, time for moment of truth. Is $L(M) = L(N)$?

- As for NFA-DFA equivalence, we'll start by proving that δ^* works "the same" for both M and N.
- Unfortunately, we have a problem. $\delta^*_N(\psi, \varepsilon) = \psi$ by definition, but $\delta^*_M(\psi, \varepsilon) = \varepsilon(\psi)$, which could be bigger.
- Hence, we will prove only that for *nonempty* strings x, $\delta^*_N(\psi, x) = \delta^*_M(\psi, x)$.
- Proceed by induction on $|x|$ as usual.
- **Bas**: when $x = a \in \Sigma$, we have that
$$\begin{aligned}
\delta^*_N(\psi, a) &= \delta^*_N(\psi, \varepsilon \cdot a) \\
&= \delta_N(\delta^*_N(\psi, \varepsilon), a) \\
&= \delta_N(\psi, a) \\
&= \varepsilon(\delta_M(\varepsilon(\psi), a)) \\
&= \varepsilon(\delta_M(\delta^*_M(\psi, \varepsilon), a)) \\
&= \delta^*_M(\psi, \varepsilon \cdot a) \\
&= \delta^*_M(\psi, a).
\end{aligned}$$

- **Ind**: when $x = ya$, $|y| > 0$, we have that

$$\begin{aligned}\delta_N^*(\psi, ya) &= \delta_N(\delta_N^*(\psi, y), a) \\ &= \delta_N(\delta_M^*(\psi, y), a) \\ &= \varepsilon(\delta_M(\varepsilon(\delta_M^*(\psi, y)), a)),\end{aligned}$$

  where second step follows from IH.

- However, notice that for all $x \in \Sigma^*$,

$$\varepsilon(\delta_M^*(\psi, x)) = \delta_M^*(\psi, x).$$

  This follows from the definition of δ_M^* and the fact that ε is idempotent.

- Conclude that

$$\begin{aligned}\delta_N^*(\psi, ya) &= \varepsilon(\delta_M(\delta_M^*(\psi, y), a)) \\ &= \delta_M^*(\psi, ya).\end{aligned}$$

Now to show that N accepts a string x iff M does.

- ($\leftarrow$) Suppose M accepts x.
- If $x = \varepsilon$, then some state of $\varepsilon(q_0)$ must be in A_M.
- In this case, our construction makes $q_0 \in A_N$, so N accepts x.
- Otherwise, we have that

$$\delta_N^*(q_0, x) = \delta_M^*(q_0, x).$$

  Since $A_M \subseteq A_N$, N will also accept x.

- ($\rightarrow$) Suppose N accepts x.
- If $x = \varepsilon$, then $q_0 \in A_N$. Hence, either $q_0 \in A_M$, or some other state of $\varepsilon(q_0)$ is in A_M. Either way, M accepts x.
- Otherwise, we again have

$$\delta_N^*(q_0, x) = \delta_M^*(q_0, x).$$

  If N accepts at some state other than q_0, then by construction, so too does M, since q_0 is the only state of N that can be in $A_N - A_M$.

- If N accepts at q_0, either $q_0 \in A_M$, or some state of $\varepsilon(q_0)$ is in A_M. In the latter case, once M reaches q_0, there is a path of null transitions that reaches an accepting state. Hence, M accepts x.
- Conclude that $L(N) = L(M)$. QED