CSE547T Class 4

Jeremy Buhler

January 30, 2017

1 Power Versus Expressiveness

- Expressiveness: how easy is it to specify a computation
- Power: is it possible to specify a computation at all?
- NFAs are more expressive than DFAs for a variety of problems.
- But, does nondeterminism confer power? Ie, are there languages NFAs can recognize that DFAs cannot?
- In short, are NFAs more powerful than DFAs?

2 NFA-DFA Equivalence

I claim that NFAs and DFAs recognize exactly the same set of languages!

- There are two directions to prove here. One is almost trivial; one is not.
- Easy: If M is a DFA, then there exists an NFA N such that $L(N) = L(M)$.
- It's easy, but let's make sure we can prove it.
- **Pf**: Let $M = (Q, \Sigma, q_0, A, \delta_M)$. Construct N as follows:
- Q, Σ, q_0, A are all the same for both M and N.
- Need to define new δ_N for N, since its range is different for NFAs than for DFAs.
- If $\delta_M(q, a) = r$, then $\delta_N(q, a) = \{r\}$.
- Do M and N accept same language?
- Surely. Can show inductively that $\delta^*_M(q, x) = r$ iff $\delta^*_N(q, x) = \{r\}$. (Use defns of δ^*!)
- It follows that M accepts precisely when N does.

If only the other direction were as trivial...

3 The Subset Construction

Claim: if N is an NFA, then there exists a DFA M such that $L(M) = L(N)$.

- What should the DFA M look like? The work done to this point should give you the hint!
- How did we define δ^* for a DFA?
- For $q \in Q$, $\delta^*(q, ya) = \delta(\delta^*(q, y), a)$.
- Now, what was our corresponding version of δ^* for NFAs?
- For $\psi \subseteq Q$, $\delta^*(\psi, ya) = \delta(\delta^*(\psi, y), a)$.
- Do these definitions look similar to you? What does that mean?
- Intuitively, we can compute what an NFA does just as we compute what a DFA does. The only difference is that instead of tracking states, we need to track (finite) *sets of states*.
- Let's build a DFA that tracks the *set of states* that N is in after reading any string x.

OK, time for formality.

- If $N = (Q, \Sigma, q_0, A, \delta_N)$, define $M = (Q', \Sigma, q_0', A', \delta_M)$.
- For each $\psi \in 2^Q$, define a state $\langle\psi\rangle \in Q'$.
- $q_0' = \langle\{q_0\}\rangle$
- $A' = \{\langle\psi\rangle \in Q' \mid \psi \cap A \neq \emptyset\}$
- Using our extended defn of δ for NFAs,

$$\delta_M(\langle\psi\rangle, a) = \langle\delta_N(\psi, a)\rangle .$$

- Above definition is called the *subset construction*.

Before we go ahead with the proof that the subset construction works, I want to do a (simple) example.

- Consider the following NFA for the language

$$L = \{x \in \{a, b\}^* \mid x \text{ ends with an } a\}.$$

- Let's build the DFA according to the subset construction.

- Interestingly, it looks like there are only two sets of states possible for this NFA – the others are not reachable from $\{q_0\}$.
- Quick check: are the DFA and NFA identical? Yes!

4 Correctness of Subset Construction

Theorem: if N is an NFA, and M is the DFA built from N via the subset construction, then $L(M) = L(N)$.

- **Pf**: Will first prove "movement equivalence": for any state $\langle\psi\rangle$ of M and any string x,
$$\delta_M^*(\langle\psi\rangle, x) = \langle\delta_N^*(\psi, x)\rangle .$$
- Proceed by induction on $|x|$.
- **Bas**: if $x = \varepsilon$, then
$$\delta_M^*(\langle\psi\rangle, \varepsilon) = \langle\psi\rangle .$$
Fortunately,
$$\langle\delta_N^*(\psi, \varepsilon)\rangle = \langle\psi\rangle .$$
- **Ind**: now let's do general case. Let $x = ya$.
- By definition of δ^* for DFAs, we have
$$\delta_M^*(\langle\psi\rangle, ya) = \delta_M(\delta_M^*(\langle\psi\rangle, y), a).$$
- Applying IH to y, the above is equal to
$$\delta_M(\langle\delta_N^*(\psi, y)\rangle, a) .$$
- Using defn of δ_M, we conclude that
$$\begin{aligned}\delta_M(\langle\delta_N^*(\psi, y)\rangle, a) &= \langle\delta_N(\delta_N^*(\psi, y), a)\rangle \\ &= \langle\delta_N^*(\psi, ya)\rangle ,\end{aligned}$$
which is what we want. Yay!

- **Corollary** to above claim:

$$\delta_M^*(\langle\{q_0\}\rangle, x) = \langle\delta_N^*(\{q_0\}, x)\rangle = \langle\delta_N^*(q_0, x)\rangle.$$

- Let $\psi_f = \delta_N^*(q_0, x)$.
- N accepts x iff $\psi_f \cap A \neq \phi$.
- But this inequality holds iff $\langle\psi_f\rangle \in A'$ by defn of A'. Moreover, $\langle\psi_f\rangle \in A'$ iff M accepts x.
- Conclude that N accepts x iff M accepts x, and we are done! QED

5 A Better Example of Subset Construction

Here is a slightly more complex example of the subset construction. Recall our NFA for accepting strings containing *aba*.

- The easiest way to actually do the subset construction is *lazily*.
- Not every possible set of NFA states yields a DFA state reachable from the start.
- To save time, we only construct *reachable* states of DFA.
- Starting from $\{q_0\}$, apply δ for all $a \in \Sigma$ to find DFA states reachable in one step.
- Recursively apply δ to these states to get states reachable in two steps. Proceed until no new states are reached.
- It is easiest to write δ for the DFA in tabular form!

Example:

One more comment on state space blowup.

- For an NFA with n states, how many subsets of states are there?
- 2^n, of course.
- Hence, (non-lazy) subset construction can yield a DFA exponentially larger than the NFA!
- Does this ever happen in practice? See practice problems.