CSE547T Class 15

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1 One Reduction to Rule Them All

I want to show you that all these special reductions we've discussed can be subsumed by one big reduction.

- Let p be a property of languages.
- (Examples: empty? finite? includes ε? is recursive?)
- Suppose that, among all RE languages, at least one language has property p, and at least one language does not.
- **Theorem** (Rice): the problem of deciding if a particular TM's language has property p is undecidable. That is,

  $$L_p = \{e(M) \mid M \text{ is a TM}, L(M) \text{ has property } p\}$$

  is not recursive.
- In other words,

All interesting properties of RE languages, and hence of sufficiently hard problems solvable on any Turing-equivalent computer, are undecidable!

Let's reduce from SA.

- Suppose to the contrary that we have a TM M_p deciding L_p.
- We will construct a TM M_{SA} to decide SA.
- Divide construction into two cases.
- **Case 1**: Suppose the empty language $\emptyset$ does *not* satisfy property p.
- Let $\hat{L}$ be any RE language that *does* satisfy property p.
- Moreover, let $M_{\hat{L}}$ be a TM accepting $\hat{L}$.
- Finally, let M_{AS} be a machine *accepting* (not deciding!) SA. We know that SA is RE, so such a machine exists.
- We construct M_{SA} as follows.
- On input x, M_{SA} first constructs the following TM $f(x) = M'$:
  - On input y, M' simulates M_{AS} on x.
  - If M_{AS} accepts x, then M' simulates $M_{\hat{L}}$ on y and accepts y iff $M_{\hat{L}}$ accepts y.
  - If M_{AS} rejects x, then M' rejects y.
- Then, M_{SA} runs M_p on $e(M')$ and returns the result of M_p.
- **Claim**: $x \in \text{SA}$ iff $f(x) \in L_p$.
- **Pf**: consider the behavior of M'.
  - If $x \in \text{SA}$, $L(M') = L(M_{\hat{L}}) = \hat{L}$.
  - If $x \notin \text{SA}$, $L(M') = \emptyset$.
  - (Indeed, M' rejects all inputs if M_{AS} rejects x, or runs forever on all inputs otherwise.)
- Since $\hat{L}$ has property p while $\emptyset$ does not, we see that M' has property p iff $x \in \text{SA}$, as desired, and the claim is proven.
- Conclude that M_{SA} decides SA, which is impossible because SA is not recursive.
- Hence L_p is not recursive either.

Now for the opposite case...

- **Case 2**: suppose $\emptyset$ satisfies property p.
- Let $\overline{p}$ be the property "not-p", and let

$$L_{\overline{p}} = \{e(M) \mid M \text{ is a TM}, L(M) \text{ has property } \overline{p}\}.$$

- Assume that L_p is recursive. Then we can decide $L_{\overline{p}}$ as follows:

- On input x, if x does not encode a TM, reject it.
- Otherwise, simulate L_p on x and accept/reject iff L_p rejects/accepts.
- But $\emptyset$ fails to satisfy $\overline{p}$, while (by assumption) some other language $\hat{L}$ fails to satisfy p and hence *does* satisfy $\overline{p}$!
- Hence, by proof of case 1, $L_{\overline{p}}$ is not recursive!
- Conclude that our assumption is false, and so L_p is not recursive.

2 Caveat on Use of Rice's Theorem

In Rice's Theorem: p is a property of *languages*, not of TMs.

- TM-specific properties that may apply to some, but not all, possible TMs for a given language may be decidable or not – Rice's Theorem does not apply!
- *Example*: does TM M ever move off its starting cell?
- Suppose M has state set Q and tape alphabet Γ.
- At most how many steps can M compute without either halting or moving off its starting cell?
- If M ever repeats a configuration, it will loop forever.
- Hence, if M runs for $|Q| \times |\Gamma|$ steps while staying in place, it will repeat and never terminate.
- Otherwise, it will either halt without moving off the starting cell, or it will move.
- Hence, this property is decidable.
- But there are also undecidable properties of TMs to which Rice's Theorem does not apply.
- *Example*: does TM M ever crash by moving off the left end of its tape?
- Reduce from HALT.
- Given TM M, modify M to create M'.
- M' is modified so that, if M would halt by accepting or by entering h_r, M' instead enters a new state q^* in which the head keeps moving left until it runs off the end.
- Observe that M' runs its head off the left side of its tape iff M halts.
- Hence, this property is undecidable.