Quiz No. 4 [Two Problems - 10 points each]

Your Name:

Problem 1 [M/G/1 queue with two job types]

Consider an M/G/1 system with two types of jobs. Type 1 jobs arrive according to a Poisson process of rate λ_1, and have service times that are exponentially distributed with mean $E[S_1] = \frac{1}{\mu_1}$. Conversely, Type 2 jobs arrive according to a Poisson process of rate λ_2, and have service times that are exponentially distributed with mean $E[S_2] = \frac{1}{\mu_2}$.

Provide expressions, functions of $\lambda_1, \lambda_2, \mu_1$ and μ_2, for the average response times, $E[T_1]$ and $E[T_2]$ of type 1 and type 2 jobs.

Write your answer in the space below

Note that the overall service time distribution is hyper-exponential, *i.e.,* exponential of rate μ_1 with probability $\frac{\lambda_1}{\lambda_1+\lambda_2}$ and exponential of rate μ_2 with probability $\frac{\lambda_2}{\lambda_1+\lambda_2}$. The arrival process is Poisson with rate $\lambda = \lambda_1 + \lambda_2$. In order to get the system's average response time, we will rely on P-K's formula, and must therefore derive the first and second moments of the service time distribution.

The average service time is

$$E[S] = \frac{\lambda_1 E[S_1]}{\lambda} + \frac{\lambda_2 E[S_2]}{\lambda} = \frac{\rho_1 + \rho_2}{\lambda} = \frac{\rho}{\lambda}$$

where $\rho_1 = \lambda_1 E[S_1] = \frac{\lambda_1}{\mu_1}$, $\rho_2 = \lambda_2 E[S_2] = \frac{\lambda_2}{\mu_2}$, and $\rho = \rho_1 + \rho_2$. Similarly, we have

$$E[S^2] = \frac{\lambda_1 E[S_1^2]}{\lambda} + \frac{\lambda_2 E[S_2^2]}{\lambda} = \frac{2}{\lambda}\left(\frac{\rho_1}{\mu_1} + \frac{\rho_2}{\mu_2}\right)$$

Both types of jobs experience the same average queueing time, which based on the P-K formula is of the form

$$E[T_Q] = \frac{1}{1-\rho}\left(\frac{\rho_1}{\mu_1} + \frac{\rho_2}{\mu_2}\right)$$

Hence, the response times for type 1 and type 2 jobs are of the form

$$\begin{aligned} E[T_1] &= E[T_Q] + E[S_1] = \frac{1}{1-\rho}\left(\frac{\rho_1}{\mu_1} + \frac{\rho_2}{\mu_2}\right) + \frac{1}{\mu_1} \\ E[T_2] &= E[T_Q] + E[S_2] = \frac{1}{1-\rho}\left(\frac{\rho_1}{\mu_1} + \frac{\rho_2}{\mu_2}\right) + \frac{1}{\mu_2} \end{aligned}$$

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Problem 2 [M/G/1/FCFS vs. M/G/1/PS for two job types]

Consider the same combination of jobs as in the previous problem, but assume now that they are directed to a server that uses the Processor Sharing (PS) service discipline. When and why do type 1 (or type 2) jobs get a better response time under PS than they do under FCFS?

Write your answer in the space below

We know that for a job of size x, the average response time under PS is of the form

$$E[T(x)]^{\text{PS}} = \frac{x}{1-\rho}$$

This means that for the two types of jobs we have

$$E[T_i]^{\text{PS}} = \frac{E[S_i]}{1-\rho}, \quad i = 1, 2$$

Without loss of generality, consider type 1 jobs to determine under which condition they experience a lower response time under PS. This requires

$$\begin{aligned}
\frac{1}{(1-\rho)\mu_1} &< \frac{1}{1-\rho}\left(\frac{\rho_1}{\mu_1} + \frac{\rho_2}{\mu_2}\right) + \frac{1}{\mu_1} \\
\frac{1}{\mu_1} &< \frac{\rho_1}{\mu_1} + \frac{\rho_2}{\mu_2} + \frac{1-\rho}{\mu_1} \\
0 &< \frac{\rho_1}{\mu_1} + \frac{\rho_2}{\mu_2} - \frac{\rho_1 + \rho_2}{\mu_1} \\
0 &< \frac{\rho_2}{\mu_2} - \frac{\rho_2}{\mu_1} = \rho_2\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right) \\
\Rightarrow \frac{1}{\mu_1} &< \frac{1}{\mu_2} \quad \text{or} \quad \mu_1 > \mu_2
\end{aligned}$$

In other words, smaller jobs will experience a smaller response time under PS than under FCFS, as we expected.