Quiz No. 3 [Two Problems, 5 points each]

Your Name:

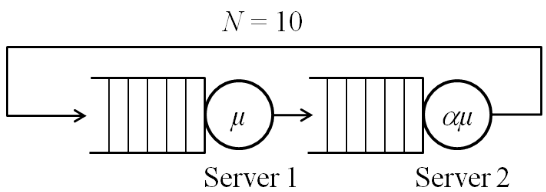

Figure 1: Closed system.

Problem 1 Consider the closed system of Fig. 1, which consists of two servers in series. The first server has a rate of μ and the second server has a rate of $\alpha\mu$. There are $N = 10$ jobs circulating in the system.

Find an expression for the value of α so that the probability that server 2 is empty is twice the probability that server 1 is empty.

We know that the state probabilities are of the form

$$\pi_{i,j} = C\left(\frac{1}{\mu}\right)^i \left(\frac{1}{\alpha\mu}\right)^j, i + j = 10,$$

where $\pi_{i,j}$ is the probability that there are i jobs at server 1 and j jobs at server 2, and C is a constant obtained from the normalization condition $\sum_{i+j=10} \pi_{i,j} = 1$.

The condition we seek is $\pi_{10,0} = 2 \cdot \pi_{0,10}$, which implies

$$C\left(\frac{1}{\mu}\right)^{10} = 2C\left(\frac{1}{\alpha\mu}\right)^{10} \quad \Rightarrow \quad \alpha = \sqrt[10]{2}$$

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Problem 2 Our goal is to compare a single server M/M/1/1 system, *i.e.,* without any waiting room, to a two servers M/M/2/2 system, also without waiting room. Specifically, assume that the server in the M/M/1/1 system has a service rate of $\alpha\mu$ and that the two servers in the M/M/2/2 system each have a service rate of μ. What value should α have as a function of $\rho = \frac{\lambda}{\mu}$, where λ is the job arrival rate to the system, to ensure that the two systems have equal blocking probabilities. Intuitively, do you expect α to be bigger or smaller than 2? Justify your intuition.

In an M/M/1/1 system with arrival rate λ and service rate $\alpha\mu$, the blocking probability is of the form:

$$P_{B1} = \frac{\rho}{\rho + \alpha}$$

while the blocking probability in the M/M/2/2 system with two servers of rate μ is of the form

$$P_{B2} = \frac{\frac{\rho^2}{2!}}{1 + \rho + \frac{\rho^2}{2!}} = \frac{\rho^2}{\rho^2 + 2\rho + 2}$$

Requiring that $P_{B1} = P_{B2}$ implies

$$\begin{aligned} \frac{\rho^2}{\rho^2 + 2\rho + 2} &= \frac{\rho}{\rho + \alpha} \\ \rho^2 + \alpha\rho &= \rho^2 + 2\rho + 2 \\ \alpha &= 2 + \frac{2}{\rho} \end{aligned}$$

which yields a value of α larger than 2. This is intuitive since from the property of the Erlang-B formula, we know that as the system size grows, for a constant load, the blocking probability decreases. Hence, additional servers are more efficient, so that a single server system needs to serve jobs faster than two individual servers to offer the same performance.