REVIEW PROBLEMS No. 3

Textbook: Problems 3.20, 3.25,

Problem S3.1: Consider a student whose life cycle consists of periods of sleep, work, and drinking coffee at a coffee house, each of duration 1 hour, but not in a deterministic pattern. Specifically, when the student wakes up after 1 hour of sleep, she goes back to sleep for another 1 hour with probability $\frac{1}{3}$, or goes to work with probability $\frac{2}{3}$. Similarly, when finishing a 1 hour work "shift", the student goes home to sleep for 1 hour with probability $\frac{1}{3}$ or heads for 1 hour at the coffee house with probability $\frac{2}{3}$. Finally, after drinking coffee for 1 hour, the student either stays at the coffee house for another hour with probability $\frac{1}{3}$ or goes back to work with probability $\frac{2}{3}$.

Assume that the student just arrived at work from home, and let T denote the number of hours before the student goes back home. What are $\mathrm{E}[T]$ and $\mathrm{Var}(T)$?

Problem S3.2: Let X be an exponentially distributed random variable with parameter λ. Consider another random variable Y that is uniformly distributed in $[0, X]$. Derive expressions, function of λ, for $\mathrm{E}[Y]$ and $\mathrm{Var}(Y)$.

Problem S3.3: Let $x(t) = \sin(\omega t + \theta)$ where ω is a constant, $0 \leq t \leq \infty$ denotes time, and θ is a real-valued random variable with known density function $f(\theta)$. Identify a function $f(\theta)$ so that $x(t)$ has the same time and ensemble averages ($x(t)$ will then also be ergodic and interestingly aperiodic).

Problem S4.4: Let $\{X_1, X_2, \ldots\}$ be a sequence of random variables such that $X_n \sim \text{Bernoulli}\left(\frac{1}{n}\right), n \in \mathbb{N}$, *i.e.,* $P(X_n = 1) = \frac{1}{n}$ and $P(X_n = 0) = 1 - \frac{1}{n}$. Prove that X_n converges to 0 in probability but not almost surely.