Quiz No. 2

Problem 1 [5 points] Consider a processing system with a single, *unit rate* server operating at a fixed clock rate. Jobs arrive at the beginning of a clock cycle with probability p, and a job in service leaves at the end of a clock cycle also with probability p. In addition, an arrival that sees the server busy (a non-idle system), leaves right away without attempting service with probability $\frac{1}{2}$. We sample the system at time slot boundaries just after any departure and before any arrival, *i.e.,* the system state does not change if there is an arrival and a departure in the same slot.

Under these assumptions, what is the actual arrival rate to the server?

Hint: View this as a DTMC and compute the probability that an arrival sees the server idle (or not).

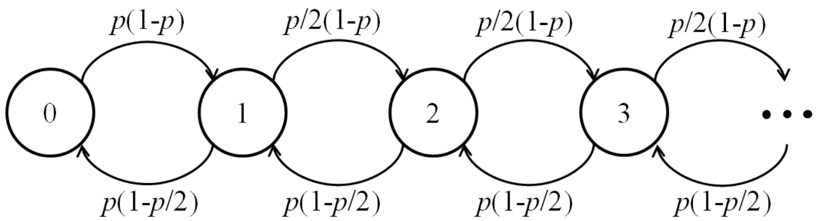

Figure 1: Markov chain representation.

Answer: The above system can be represented using the Markov chain of Fig. 1, which in turn gives rise to the following balance equations:

$$
\begin{aligned}
p(1-p)\pi_0 &= p\left(1-\frac{p}{2}\right)\pi_1 \Rightarrow \pi_1 = \frac{2(1-p)}{2-p}\cdot\pi_0 \\
\frac{p}{2}(1-p)\pi_i &= p\left(1-\frac{p}{2}\right)\pi_{i+1} \\
\Rightarrow \pi_{i+1} &= \frac{1-p}{2-p}\pi_i \Rightarrow \pi_i = \alpha^{i-1}\pi_1\,, i\geq 1 \text{ where } \alpha = \frac{1-p}{2-p}
\end{aligned}
$$

Combining this with the normalization condition $\sum_{i=0}^{\infty}\pi_i = 1$ gives

$$
\begin{aligned}
\pi_0 + \pi_1\sum_{i=0}^{\infty}\alpha^i &= \pi_0\left(1+\frac{2(1-p)}{2-p}\cdot\frac{1}{1-\alpha}\right) = (3-2p)\pi_0 = 1 \\
\Rightarrow \pi_0 &= \frac{1}{3-2p}
\end{aligned}
$$

where we have used the fact that $1-\alpha = \frac{1}{2-p}$.

Hence, with probability $\frac{1}{3-2p}$ the arrival rate is p and with probability $1-\pi_0 = \frac{2(1-p)}{3-2p}$ is its $\frac{p}{2}$. This combines to give an average arrival rate to the server of

$$
\lambda = \frac{1}{3-2p}\cdot p + \frac{2(1-p)}{3-2p}\cdot\frac{p}{2} = \frac{p(2-p)}{3-2p}
$$

Problem 2 [5 points] Jobs arrive to a computer system according to a Poisson process at a rate of $1,000$ jobs/sec. You are being told that only one job arrived in the last 10msec.

1. **[2 points]** What is the probability of this event?

Answer: Given a Poisson process of rate $\lambda = 1,000$, the probability P_1 of one arrival in an interval of length 10msec is given by

$$P_1 = e^{-1,000 \times 0.01} \cdot \frac{1,000 \times 0.01}{1!} \approx 4.54 \times 10^{-4}$$

2. **[3 points]** What is the probability that the job arrived in the last 5msec?
Answer: Given that we have one arrival from a Poisson process in a time inteval of length t, we know (see Theorem 11.9) that the event is equally likely to have occurred anywhere in $[0, t]$. This then implies that the probability that the job arrived in the last 5msec is simply $\frac{1}{2}$.