Quiz No. 3 [One Problem – 2 questions]

Your Name:

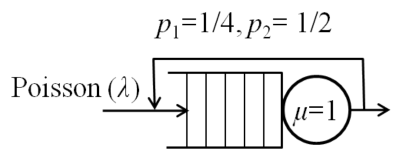

Figure 1: Classed feedback system.

Problem 1 Consider the system of Fig. 1. Jobs arrive to the system according to a Poisson process of rate λ. The server has unit service rate $\mu = 1$ job/sec. Jobs that complete their first service return for a second service with probability $p_1 = \frac{1}{4}$ and leave with probability $1 - p_1 = \frac{3}{4}$. Jobs that have returned for a second service are, however, now more likely to again come back for another service and return with probability $p_2 = \frac{1}{2}$ (and leave with probability$1 - p_2 = \frac{1}{2}$). This is an instance of a classed network.

1. What is the maximum arrival rate λ^* that keeps the system stable?

Write your answer in the space below
We have the following routing probabilities for jobs that finish service:

$$p_{1,1} = 0, p_{1,2} = \frac{1}{4}, p_{1,out} = \frac{3}{4} \quad \text{and} \quad p_{2,1} = 0, p_{2,2} = \frac{1}{2}, p_{2,out} = \frac{1}{2}$$

and therefore the following equations for the arrival rates of each class:

$$\begin{aligned} \lambda_1 &= \lambda \\ \lambda_2 &= \lambda_1 \cdot \frac{1}{4} + \lambda_2 \cdot \frac{1}{2} = \frac{\lambda}{2} \end{aligned}$$

This yields a total arrival rate $\lambda_{\text{tot}} = \lambda_1 + \lambda_2 = \frac{3\lambda}{2}$. For the system to be stable, we need $\lambda_{\text{tot}} < \mu = 1$. This implies

$$\lambda_{\text{tot}} < \lambda^* = \frac{2}{3}$$

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2. Assuming $\lambda < \lambda^*$, what are the respective average system times, $E[T_O]$ and $E[T_R]$ of original and return jobs, respectively. In other words, how long does a job that only requires one service time stays in the system, versus one that requires multiple service times? Note that the system time of a job that requires multiple service times will include its first service time.
Hint: The easiest way is to consider the average number of visits that each type of job makes to the server.

Write your answer in the space below
The system load is $\rho = \frac{3\lambda}{2\mu}$, and since the system behaves as an M/M/1 queue with load ρ, we have

$$E[N] = \frac{\frac{3\lambda}{2\mu}}{1 - \frac{3\lambda}{2\mu}} = \frac{3\lambda}{2\mu - 3\lambda}$$

so that the average service time $E[T]$ is of the form

$$E[T] = \frac{E[N]}{\frac{3\lambda}{2}} = \frac{2}{2\mu - 3\lambda}$$

Original jobs visit the server only once, while return jobs visit it on average a total of $1 + 2 = 3$ times. Hence, we have

$$E[T_O] = E[T] = \frac{2}{2\mu - 3\lambda} \quad \text{and} \quad E[T_R] = \frac{6}{2\mu - 3\lambda}$$

As a simple check, we know that $\frac{3}{4}$th of the jobs are original jobs and $\frac{1}{4}$th of the jobs are return jobs. This means that

$$\begin{aligned} E[T_{\text{tot}}] &= \frac{3}{4}E[T_O] + \frac{1}{4}E[T_R] \\ &= \frac{3}{4}\frac{2}{2\mu - 3\lambda} + \frac{1}{4}\frac{6}{2\mu - 3\lambda} \\ &= \frac{3}{2\mu - 3\lambda} \end{aligned}$$

Applying Little's Law to the overall system, we get

$$E[T_{\text{tot}}] = \frac{E[N]}{\lambda} = \frac{3}{2\mu - 3\lambda}$$

which, fotunately for us, is the same answer.