This is an open-book, open-notes exam. A memory sheet is allowed, but must be no bigger than 8.5" by 5.5" in size (one half of a standard typewriter page); writing on both sides is allowed. Use of a calculator is allowed, but numeric answers may be left unsimplified.

Please work 5 of the 6 problems below. Answers should be written on clean paper, with each problem beginning on a new page, and work presented on one side only of each page. The problems vary in difficulty and time required to answer; the exam will end at 2:30pm, so plan your time accordingly. Be sure to write your name on each page you submit.

1. A camera has a sensor chip whose active area is 4mm by 3mm in size. The active area is fully covered with pixel sensing elements arranged as a rectangular array of elements 5μm square (5μm = 5.0 x 10^-6 meters). The camera lens has a focal length of 20mm. (1mm = 1.0 x 10^-3 meters) **Be careful with units!**

a) Give the MxN pixel dimensions of the chip array.

\[ \frac{4\text{mm}}{0.005\text{mm}} = 800 \quad \frac{3\text{mm}}{0.005\text{mm}} = 600 \]

b) Sketch a pinhole camera model diagram showing the Y and Z axes of the camera coordinate frame and the image plane. (The X axis will be pointing into the paper.) Label the model components:

- principal point
- principal axis
- sensor plane
- sensor chip (cross section)
- focal point

![Camera Model Diagram](image)

C) State the size in pixels on the sensor chip of the image of a tree positioned 200 meters in front of the camera and 10 meters tall. Assume the camera line of sight (principal axis) is pointed at the base of the tree, and the camera Y axis points up, in the vertical direction.

\[ \frac{h}{200} = \frac{10}{200} \quad h = 1\text{mm} = \frac{1\text{mm}}{0.005\text{mm}} = 200\text{ pixels} \]

D) Write a general equation relating distances to objects and their heights to the projected image height on the sensor plane. Show a diagram with the variables used.

\[ \frac{h}{f} = -\frac{H}{R-f} \]
2. The image below is rotated, but it is necessary to measure various points relative to the standard reference frame C shown on the image, in order to validate the layout of components. An image reference frame I is given by the frame around the image, with origin in the lower left corner, X1 axis along rows, and Y1 axis up columns.

Three points have been measured on the reference frame C: the origin at \(<226,116>_C\), a point on the Xc axis at \(<335,76>_C\), and a point on the Yc axis at \(<265,227>_C\). Note that the measured points are in I frame coordinates. Assume both frames I and C have the same scale, so 1 unit in I is the same as 1 unit in C. Calculations to 0.1 pixel and 0.001 in angle measures are OK.

![Circuit Board Diagram](image)

\[
\begin{align*}
\mathbf{u} &= (109,-40)^T \\
\mathbf{U} &= (0.939,-0.345)^T \\
\mathbf{v} &= (39,111)^T \\
\mathbf{V} &= (0.731,0.943)^T
\end{align*}
\]

a) Write a homogenous coordinate transform matrix to map points measured in frame C coordinates in frame I coordinates.

\[
T_{IC} = \begin{pmatrix}
0.939 & 0.345 & 226 \\
-0.345 & 0.943 & 116 \\
0 & 0 & 1
\end{pmatrix}
\]

b) Apply the transform to the points \(<1,0>_C\) and \(<0,1>_C\).

c) Write a coordinate transform matrix to map point in image frame I coordinates into coordinates for C frame. (This transform will allow image point coordinates to be transformed so object placement relative to frame C can be evaluated.)

\[
T_{CI} = \begin{pmatrix}
0.939 & -0.345 & -172.2 \\
0.345 & 0.943 & -356.4 \\
0 & 0 & 1
\end{pmatrix}
\]

d) Determine the C frame coordinates of the (white) hole in the upper left center of the circuit board, at \(<161,291>_I\).
3. An image of 512x512 pixels has a pattern of bright bars 15 pixels wide and spaced 15 rows apart, with straight sides and oriented horizontally in the image.

a) Give the equation relating spatial wavelength $\Delta x$ and discrete Fourier frequency $\Delta u$ in an image row of size $M$ pixels. $\Delta u \cdot \Delta x = M$ or $u \cdot \Delta x = M$

b) Give the spatial frequency (not the wavelength) of the bar pattern.
$\Delta x = 15 / 15 = 1 \text{ cm} \quad M = 512 \quad \Delta u = 512 / 30 = 17.1$

c) Give the 2 locations in $u,v$ frequency space of the maximal coefficients that will be produced by the Fourier transform of this image.
Bars are horizontal, so $u = 0, \quad v = \pm 17.1$ \quad $\langle u,v \rangle = \langle 0,17 \rangle$ and $\langle 0,-17 \rangle$

d) The bar pattern produces vertical stripes of high magnitude Fourier transform coefficients. If the bar image is rotated clockwise by 27 degrees, tell what happens to the stripes in its new Fourier transform.

4. Below is the 2-D discrete Fourier transform.

$$ F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-i2\pi (ux/M + vy/N)} $$

a) Rearrange this equation to show its separable form.

b) Define the continuous impulse function.

c) State the sifting property for the 1-D continuous impulse for function $f(x)$.

d) State what it means for "$f(x) \star g(x)$" and "$F(u)G(u)$" to be a "conjugate pair", and give the name of the theorem of which this pair is one part.

4c) $f(x_o) \star g(x) = \int_{-\infty}^{\infty} f(x) g(x-x_o) dx$

4d) A conjugate pair is any function and its Fourier transform. Here $\mathcal{F}(f(x) \star g(x)) = F(u) \cdot G(u)$ so $f(x) \star g(x)$ and $F(u) \cdot G(u)$ are a conjugate pair. They are one half of the Convolution Theorem.
5. Examine the Fourier transform shown below of an image with periodic noise, where the peaks shown are at <231,300> and <281,212>. The image center is at <256,256>, and the <0,0> pixel coordinate is in the upper-left corner.

The peaks points are at <25,44> and <25, -44> relative to the center at <256,256>.

A ideal band-reject filter with square blocks of width W centered on the 2 peaks is given by

\[ H(u,v) =
\begin{cases}
0 & \text{if } -5W \leq u \leq 5W \text{ and } -4W \leq v \leq 4W \\
0 & \text{if } 25 \leq u \leq 25 \text{ and } -4W \leq v \leq 4W \\
1 & \text{otherwise}
\end{cases}
\]

a) Write an equation for an ideal band-reject filter H(u,v) for this image, to suppress the noise indicated by those peaks. (Above right, H(u,v) with square peak blocking.)

b) Describe the noise that might be seen in the unfiltered image, and which would be related to these peaks. Distance from center to peaks is 50.6 in this 512 x 512 image, so this could represent a sinusoidal noise of spatial wavelength \( \Delta x = 50.6/512 = 0.1 \text{ pix.} \)

The pattern is at \( \theta = \tan^{-1}(44/25) = 59.4^\circ \) above the u-axis so the spatial waves are perpendicular to a line at angle \( \theta \) to the x-axis.

c) Let \( G(u,v) \) be the Fourier transform shown, and let \( H(u,v) \) be the band-reject filter.

Write the equation showing how the restored image \( \hat{f}(x,y) \) is obtained.

\[ \hat{f}(x,y) = \mathcal{F}^{-1}(H(u,v) \cdot G(u,v)) \]

d) Suppose the size of an image is MxN pixels, but M \neq N, and neither M nor N is a power of 2. Explain how this condition is overcome so that the FFT (fast Fourier transform) can be used to restore the image. Pad the image in X or Y with null pixels, so that each dimension becomes a power of 2. (zero value)

e) If two images have dimensions MxN and CxR, explain how must these images be modified to perform their convolution. Give formulas for any parameters that must be calculated. The images must be padded with zero valued pixels to a common size large enough so that if even a single pixel of either unpadded image overlays any of the other unpadded image, both unpadded parts of the padded images will fully overlay parts of the padded images. Pad both images to size \( P \times Q \), where

\[
P > M + C - 1 \\
Q > N + R - 1
\]
6. The image noise model may be given by an equation \( g(x, y) = h(x, y) \star f(x, y) + \eta(x, y) \).

a) Define each term in the model equation.

b) Suppose an estimate of the degradation transform \( H(u, v) \) can be produced. Show how that estimate might be used to produce an estimate of the restored image.

c) Describe how the point spread function of the transform \( H(u, v) \) might be obtained for an imaging system that is available for testing.

d) Assume the noise function has some high value coefficients in its Fourier transform, \( N(u, v) \). Show a formula that indicates how that situation may increase error in the restored image, and state conditions under which that can happen.

e) Of the terms in the noise model, \( g(x, y) \), \( h(x, y) \), \( f(x, y) \), \( \eta(x, y) \), state which terms
   i) are always known, \( g(x, y) \)
   ii) may be discovered by experimentation, and
   iii) which may sometimes be assumed zero.

6a) \( g(x, y) \) — the degraded image

6b) \( h(x, y) \) — the degrading transform in spatial coordinates

6c) \( f(x, y) \) — the original, undegraded image

6d) \( \eta(x, y) \) — additive noise, often random

6e) \( \hat{f}(x, y) = \frac{1}{H(u, v)} \left( \hat{g}(u, v) \right) \) where \( \hat{F}(u, v) = \mathcal{F}(g(x, y)), \hat{H}(u, v) = \mathcal{F}(h(x, y)), \hat{N}(u, v) = \mathcal{F}(\eta(x, y)) \)

6f) Arrange to capture an image of an isolated, point-like, bright source, which simulates an input impulse. That captured image is the point spread function for the system. Images \( g(x, y) \) produced by that system are approximated by convolving the PSF with an un-degraded input image:

6g) In the Fourier domain, \( G(u, v) = F(u, v) + \frac{N(u, v)}{H(u, v)} \), so if \( N(u, v) \) has large coefficients, the noise error in \( G(u, v) \) may be increased. If the degradation transform \( H(u, v) \) has very small or near-zero value coefficients, the quotient \( \frac{N(u, v)}{H(u, v)} \) may yield large value coefficient error into \( F(u, v) \).