Commonly Used Formulae

Series

1. Arithmetic Series
\[ \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \]

2. Sum of squares
\[ \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \]

3. Sum of cubes
\[ \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \]

4. Geometric series
\[ \sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1} \]

5. Geometric series with \(|x| < 1\)
\[ \sum_{k=0}^{\infty} x^k = \frac{1}{1 - x} \]

6. Harmonic numbers
\[ H_n = \sum_{i=1}^{n} \frac{1}{i} = \ln n + \Theta(1) \]

Probability

More background on probability in Appendix C.2 of the textbook.

1. Definition of expectation.
\[ E[X] = \sum x \Pr\{X = x\} \]
2. Linearity of Expectation

\[ E[X_1 + X_2] = E[X_1] + E[X_2] \]

3. If \( X_1 \) and \( X_2 \) are independent,

\[ E[X_1X_2] = E[X_1]E[X_2] \]

4. Markov’s Inequality

\[ \Pr\{X \geq a\} \leq \frac{E[X]}{a} \]

5. If \( X \) is an indicator random variable

\[ E[X] = \Pr\{X = 1\} \]

**Other Useful Inequalities**

1. Stirling’s approximation

\[ \ln n! \leq n \ln n - n + O(\ln n) \]

A more precisely *asymptotic* variant (valid for large \( n \)) is:

\[ n! \approx \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \]

If you want something valid for all \( n \), then you can use

\[ \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \leq n! \leq e\sqrt{n} \left( \frac{n}{e} \right)^n \]

2. \( \binom{n}{k} \) notation (called \( n \) choose \( k \)): Given a set of \( n \) distinct elements, \( \binom{n}{k} \) represents the number of distinct \( k \) element subsets of this set. In other words, it is the number of ways of choosing \( k \) elements from a given set of \( n \) distinct elements. If \( n \) and \( k \) are non-negative and \( k \leq n \), then

\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \]

3. For \( k \leq n \),

\[ \binom{n}{k} \leq \frac{n^k}{k!} \]

4. For \( k \leq n \),

\[ \left( \frac{n}{k} \right)^k \leq \binom{n}{k} \leq \left( \frac{ne}{k} \right)^k \]
5. Taylor series expansion for $e^x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

6. Taylor series expansion for $\ln(1 + x)$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

7. Useful inequality for small $t$.

$$e^t \geq 1 + t$$

8. For positive $t, n$

$$\left( 1 + \frac{t}{n} \right)^n \leq e^t \leq \left( 1 + \frac{t}{n} \right)^{n+\frac{t}{2}}$$

The above inequality also holds for negative $t$ if $n \geq 1$ and $|t| \leq n$.

**Asymptotic Notation**

For any $\varepsilon > 0$ and any $k$, $n^\varepsilon = \omega(\log^k n)$.