Lecture 13: Weighted Shortest Paths

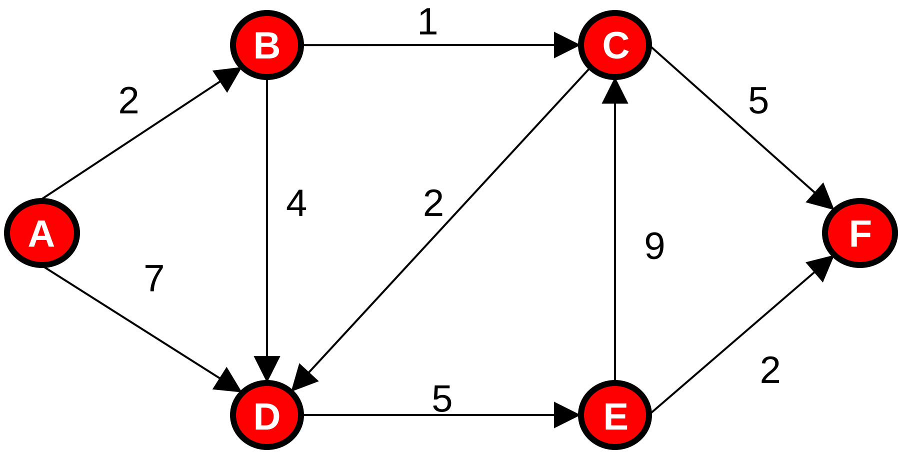
These slides include material originally prepared by Dr. Ron Cytron, Dr. Jeremy Buhler, and Dr. Steve Cole.

Announcements

- Lab 13 released tomorrow – Dijkstra’s algorithm
- Studio 13 Thursday
- Exam 3 May 1st 10 am – 12 pm

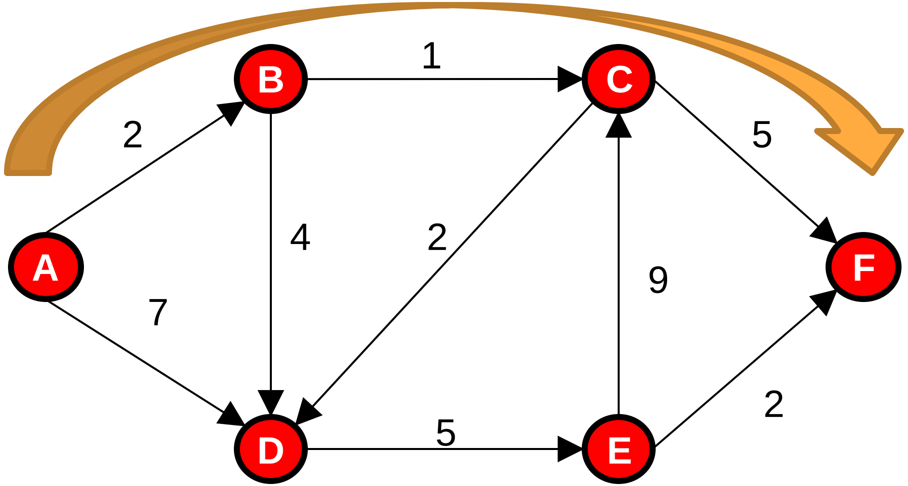
Adding Weights to Graphs

- A weighted graph assigns to each edge e a real-valued **weight** w(e)
- For today, we will assume that **w(e) ≥ 0**.



Path Lengths

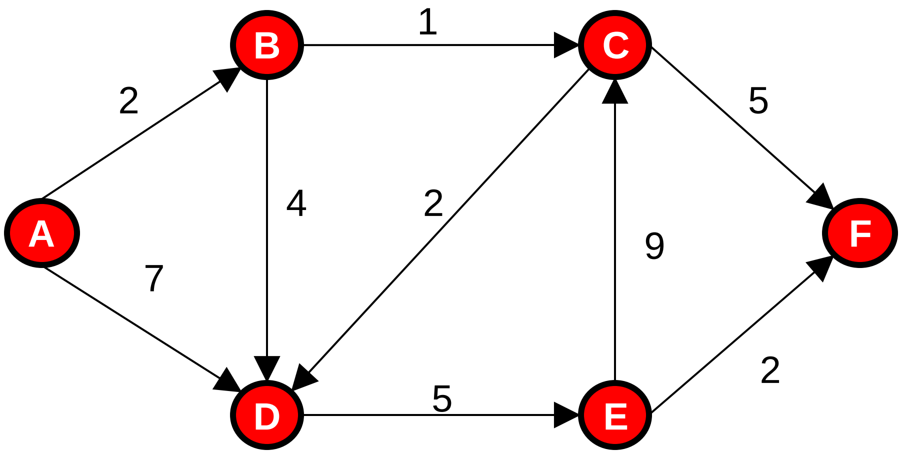
- The **length** (or total weight) of a path is the sum of its edges' weights



Total weight (ABCF)
= 2+1+5=8

Weighted Shortest Paths

- **Problem**: given a starting vertex v, *find path of* ***least total weight*** from v to each other vertex in the graph.



Weighted Shortest Paths

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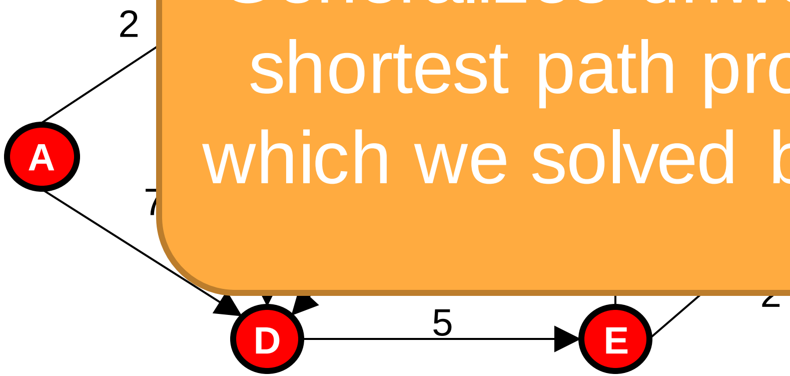
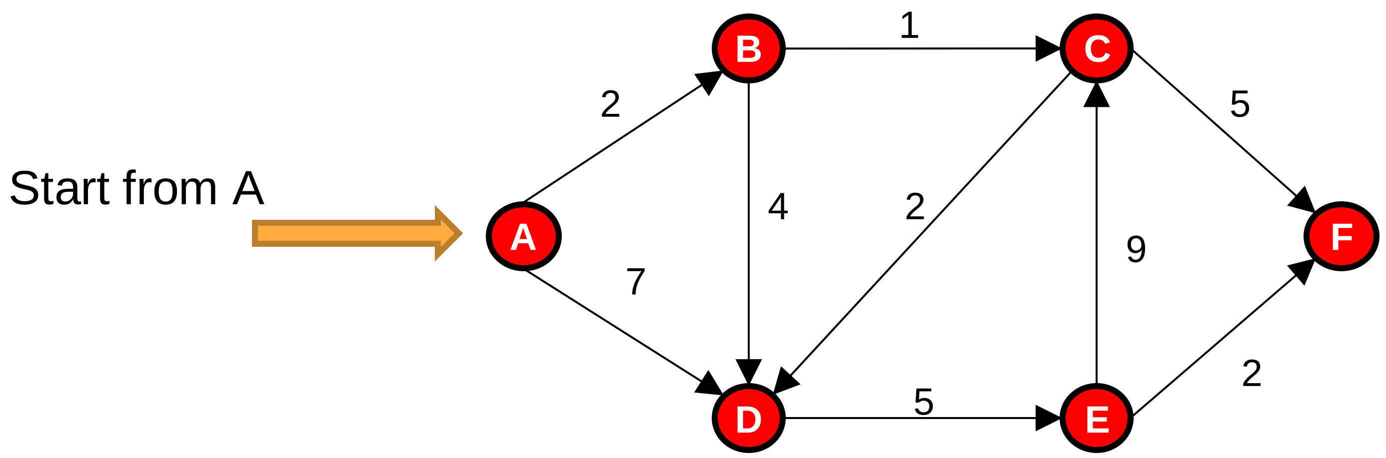


Generalizes unweighted shortest path problem, which we solved by BFS.

Weighted Shortest Paths

- Given a starting vertex v, find path of least total weight D(v,u) from v to each other vertex u in the graph.



Weighted Shortest Paths

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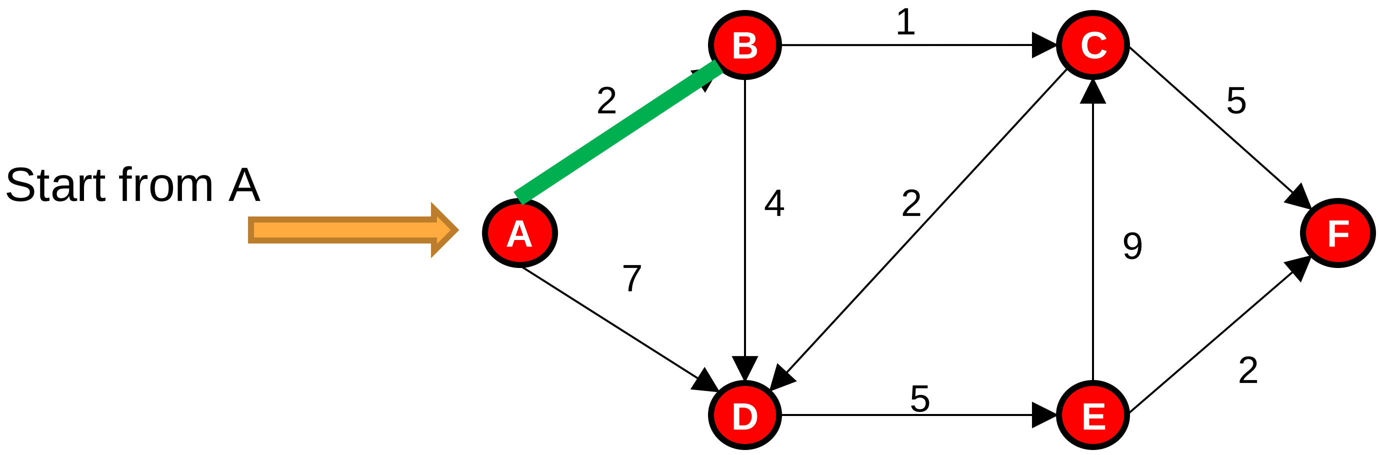
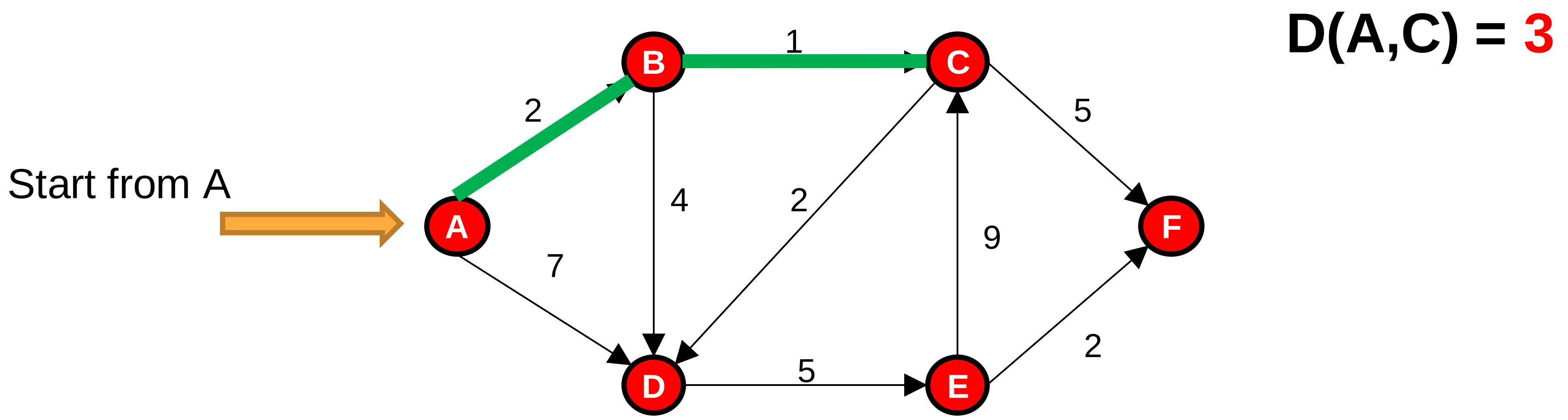


D(A,B) = 2

Weighted Shortest Paths

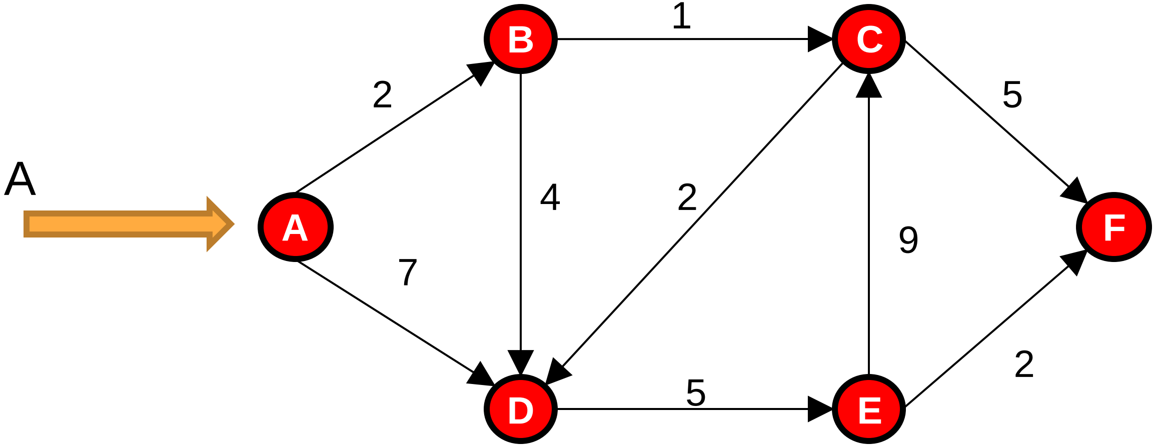
- Given a starting vertex v, find path of least total weight D(v,u) from v to each other vertex u in the graph.



Weighted Shortest Paths

- Given a starting vertex v, find path of least total weight D(v,u) from v to each other vertex u in the graph.

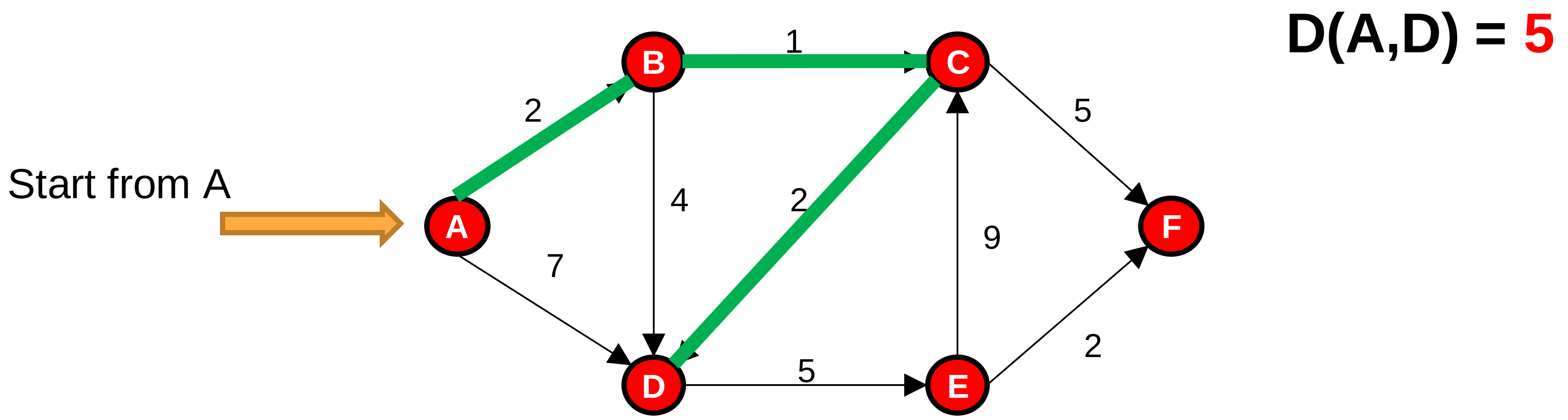
Start from A



D(A,D) = ???

Weighted Shortest Paths

- Given a starting vertex v, find path of least total weight D(v,u) from v to each other vertex u in the graph.

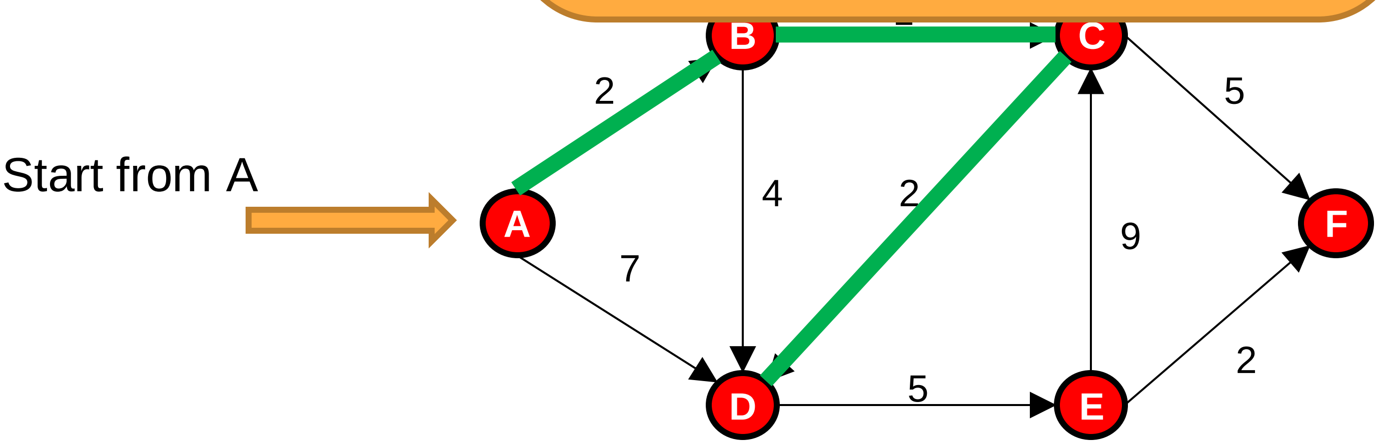


Weighted Sh

- Given a starting ... nt D(v,u) from v to each other ve

Weighted shortest path to a vertex may not be one with fewest edges.

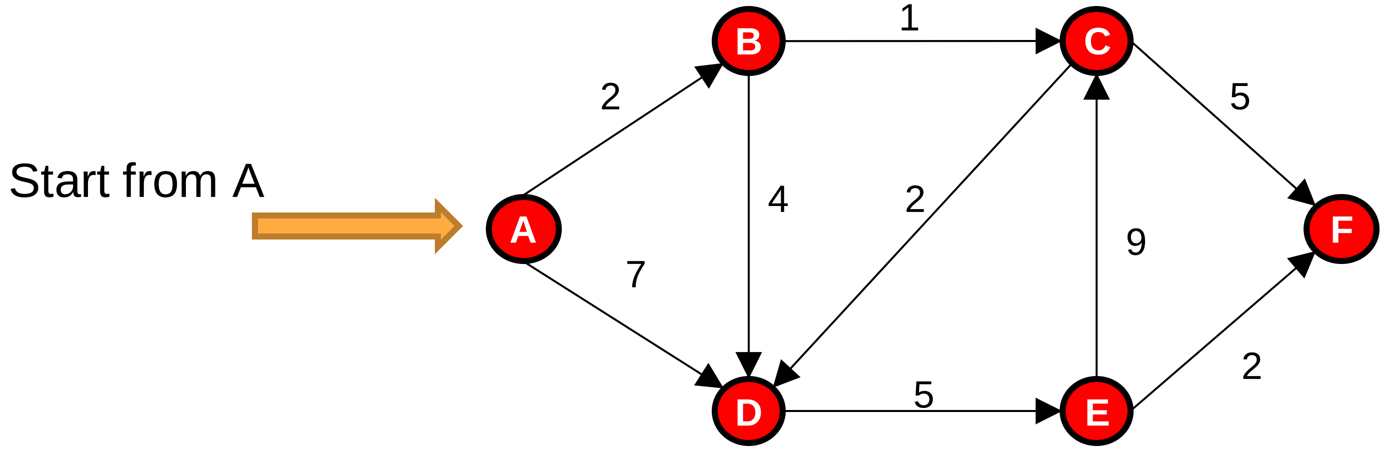
D(A,D) = 5



Weighted Shortest Paths

- Given a starting vertex v, find path of least total weight D(v,u) from v to each other vertex u in the graph.

D(A,F) = ???



Weighted Shortest Paths

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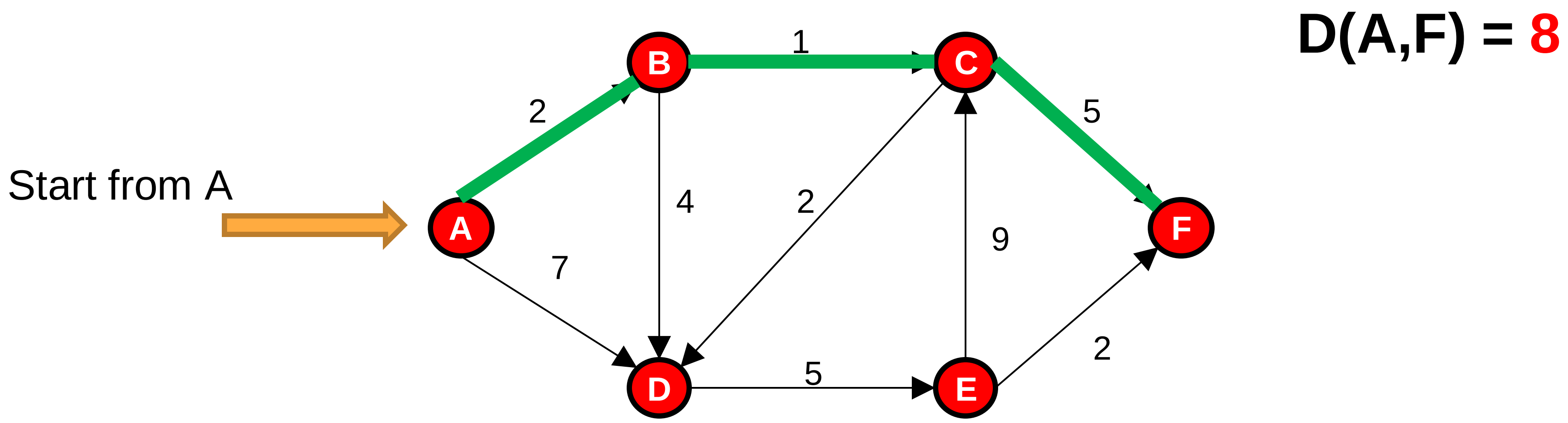
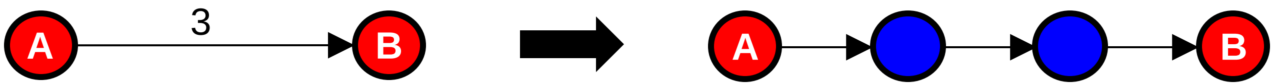


Why Solve Weighted Shortest Paths?

- Road map with distances between cities – shortest route
- Routing network with cost to each hop – cheapest way to send data
- State-space search in AI with action costs (A* search)
- …

How Can We Solve Weighted Shortest Paths?

- Could reduce to unweighted problem:



- Apply to every edge; use BFS on resulting graph
- Only works if weights are integers
- Would be very expensive for graphs with large weights

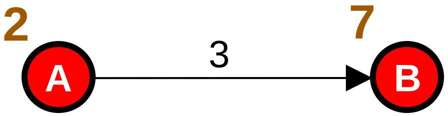
Alternate Strategy – "Relaxation"

- Explore graph while maintaining, for each vertex v, length of shortest path to v seen *so far*. Store this shortest path estimate as **v.dist**.

- Whenever we follow an edge (v,u), check whether

$$v.dist + w(v,u) < u.dist$$

- If so, *we've found a new, shorter path to u via v.*



Old B.dist = 7

- Update v.dist with new estimate and continue

Alternate Strategy – “Relaxation”

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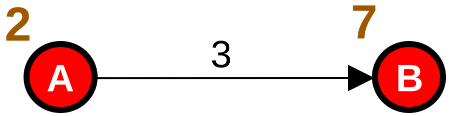

2+3=5 < 7

- Update v.dist with new estimate and continue

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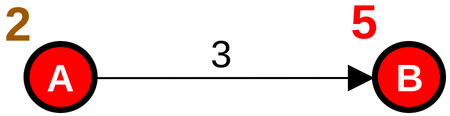


B.dist ← 5

- Update v.dist with new estimate and continue

Who Gets to Relax, When?

- Need to decide which vertices to relax at each step.
- Also, need to know when we are done!
- **Proposal** (*E. Dijkstra* *): at each step, explore edges out of vertex v with *smallest* v.dist, and relax all its adjacent vertices.
- Stop when each vertex has had its outgoing edges explored once.

* Fun fact: font on the title slide is Dijkstra's handwriting!

Dijkstra’s Shortest Path Algorithm

- starting vertex v gets v.dist ← 0; all other u get u.dist ← ∞
- mark all vertices as unfinished

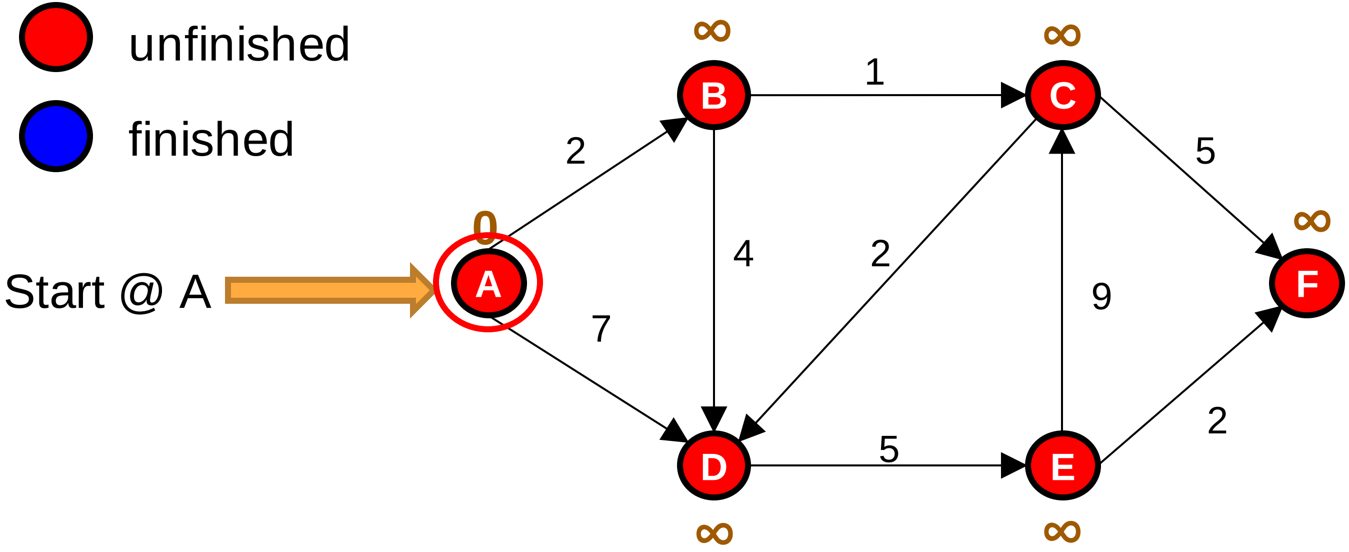
- while (*any vertex unfinished*)
-     v ← unfinished vertex with smallest v.dist
-     for each edge (v,u)
-         if (v.dist + w(u,v) < u.dist)
-             u.dist ← v.dist + w(u,v)   // relax!
-     mark v finished

Example of Dijkstra’s Algorithm

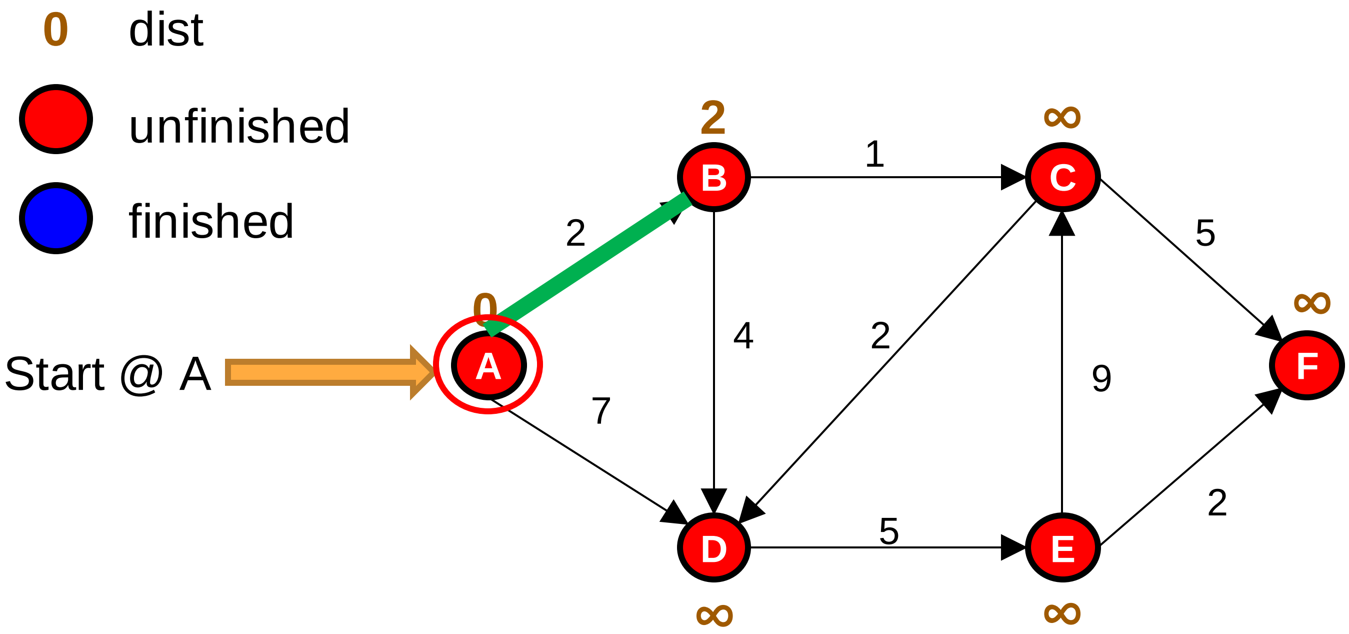
0 dist

unfinished

finished



Example of Dijkstra’s Algorithm



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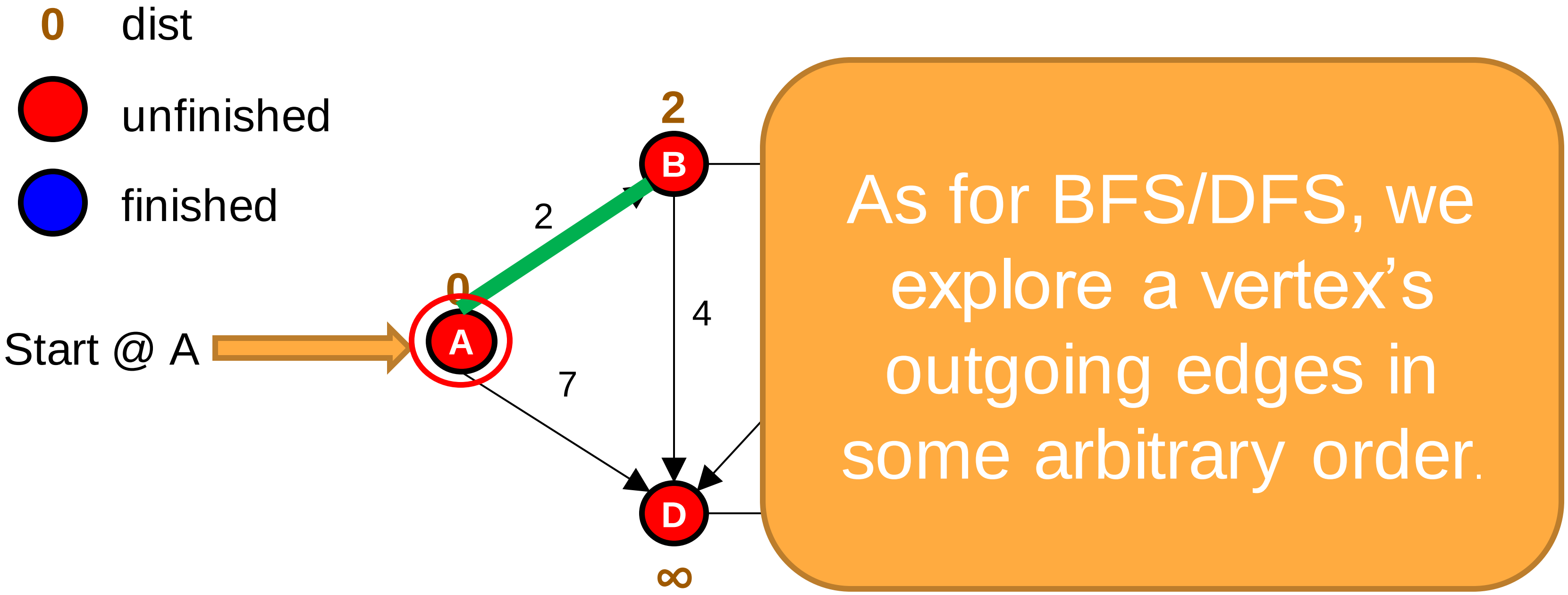

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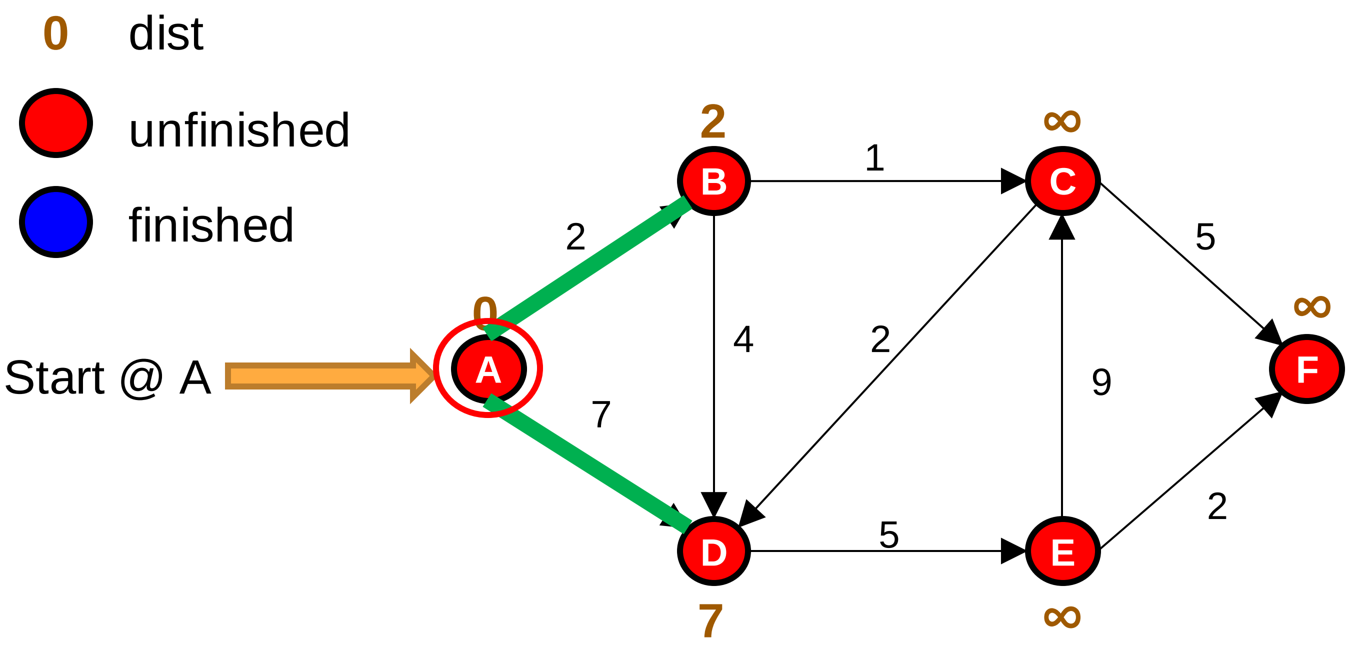

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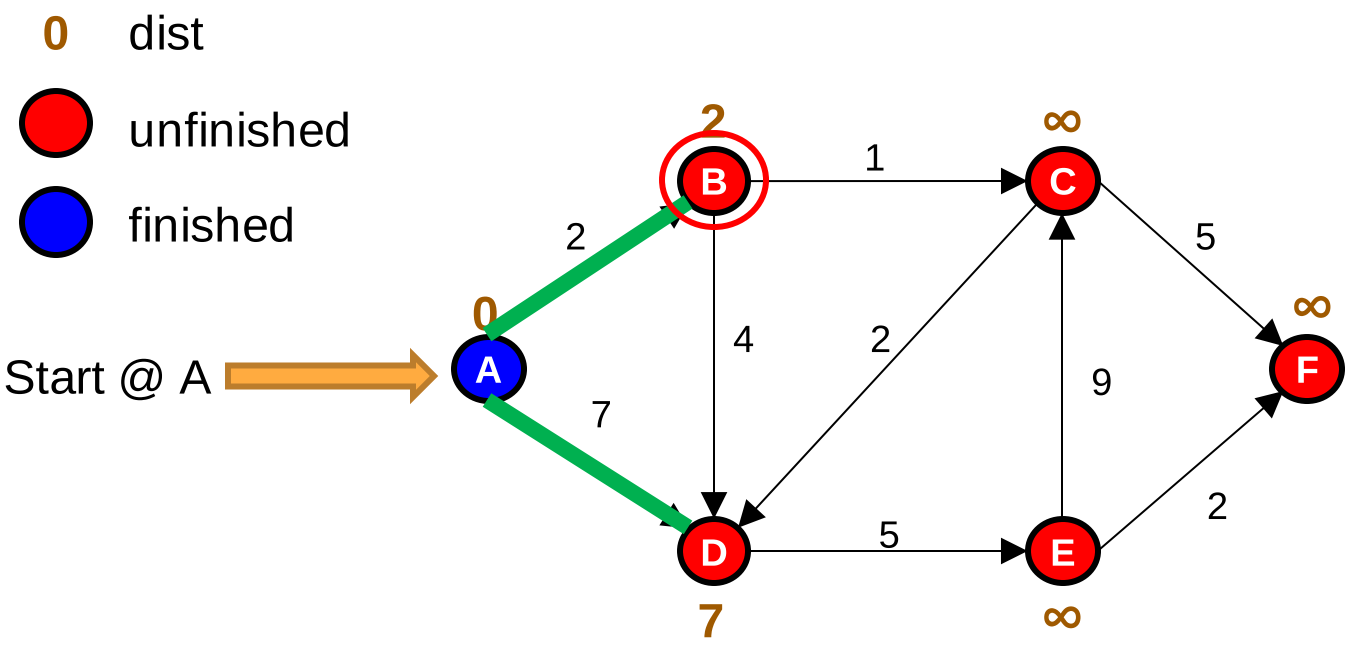

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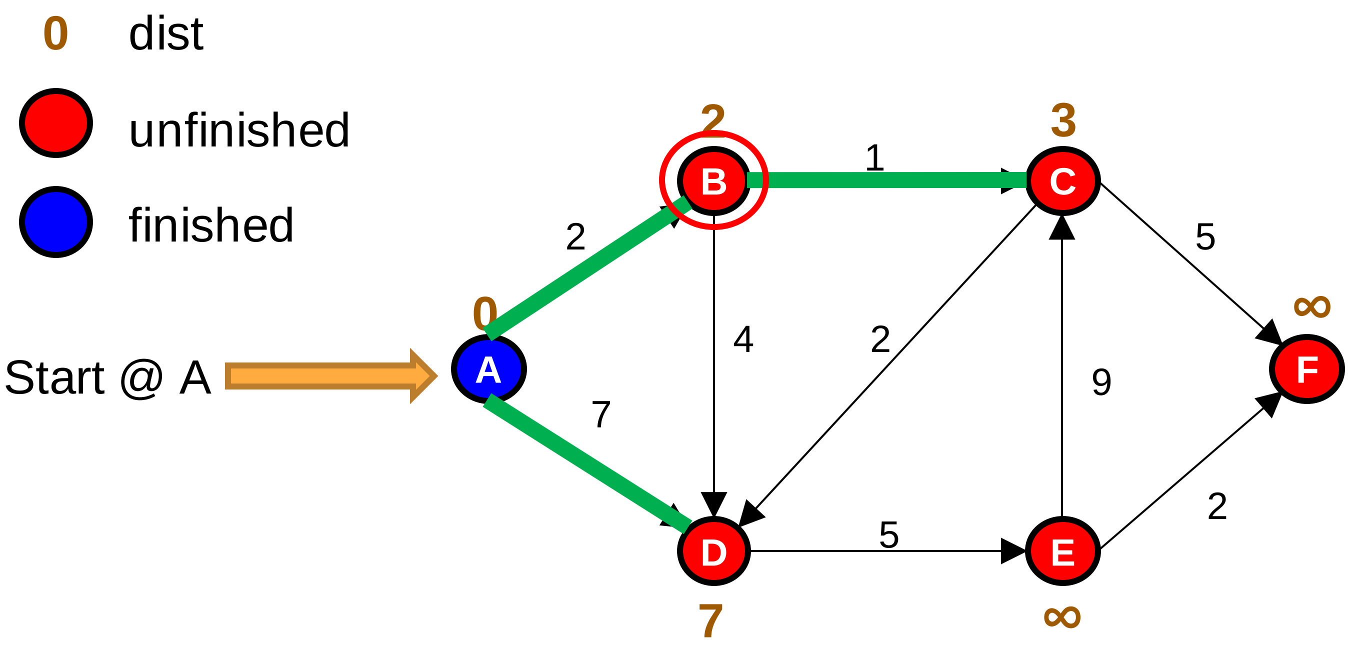
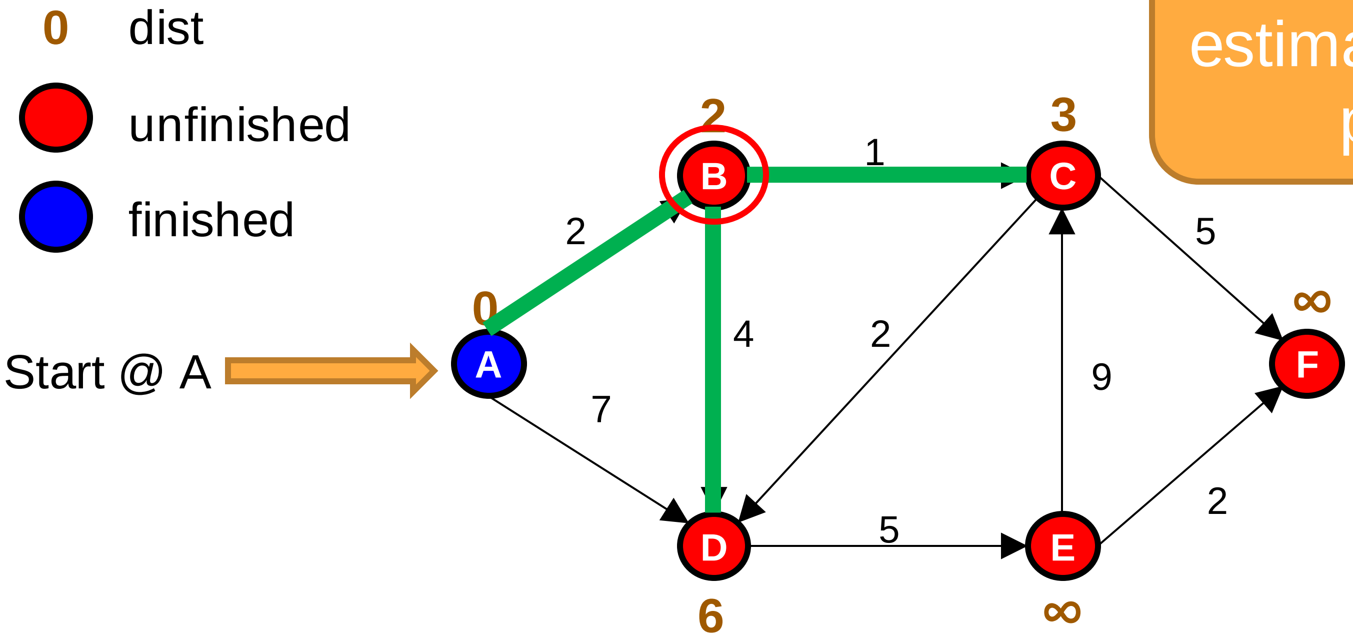


Example of Dijkstra's Algorithm

Relaxation can change estimated distance and parent edge!



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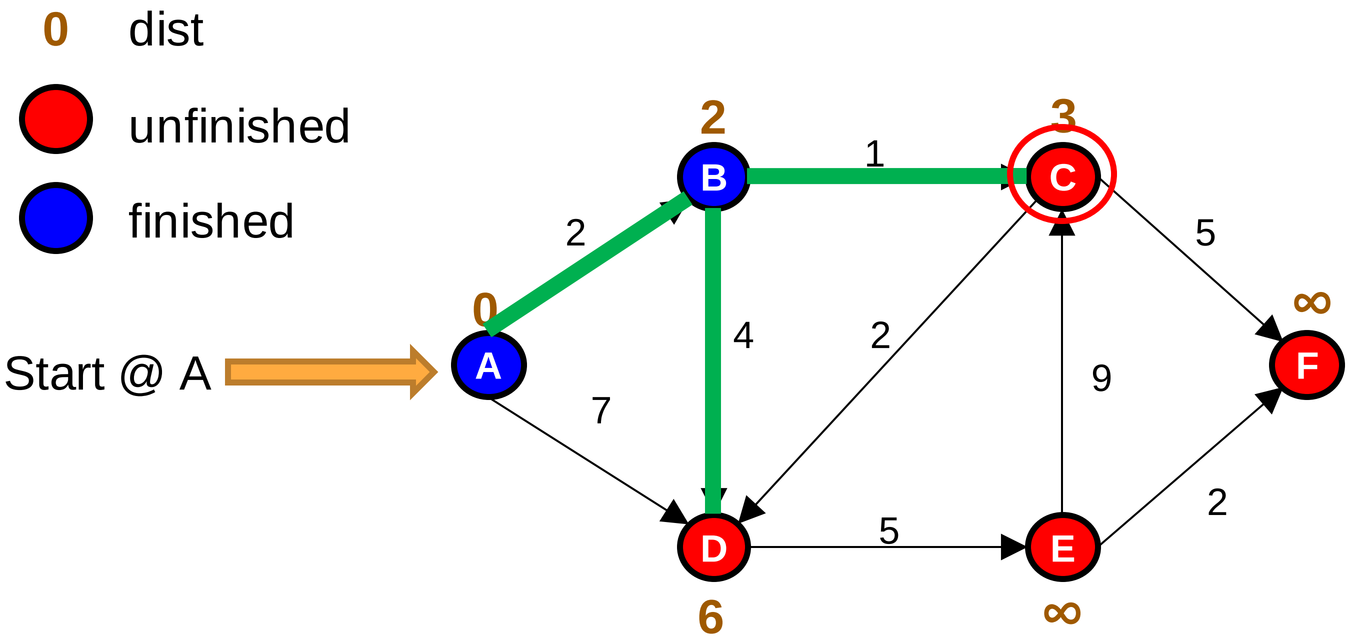
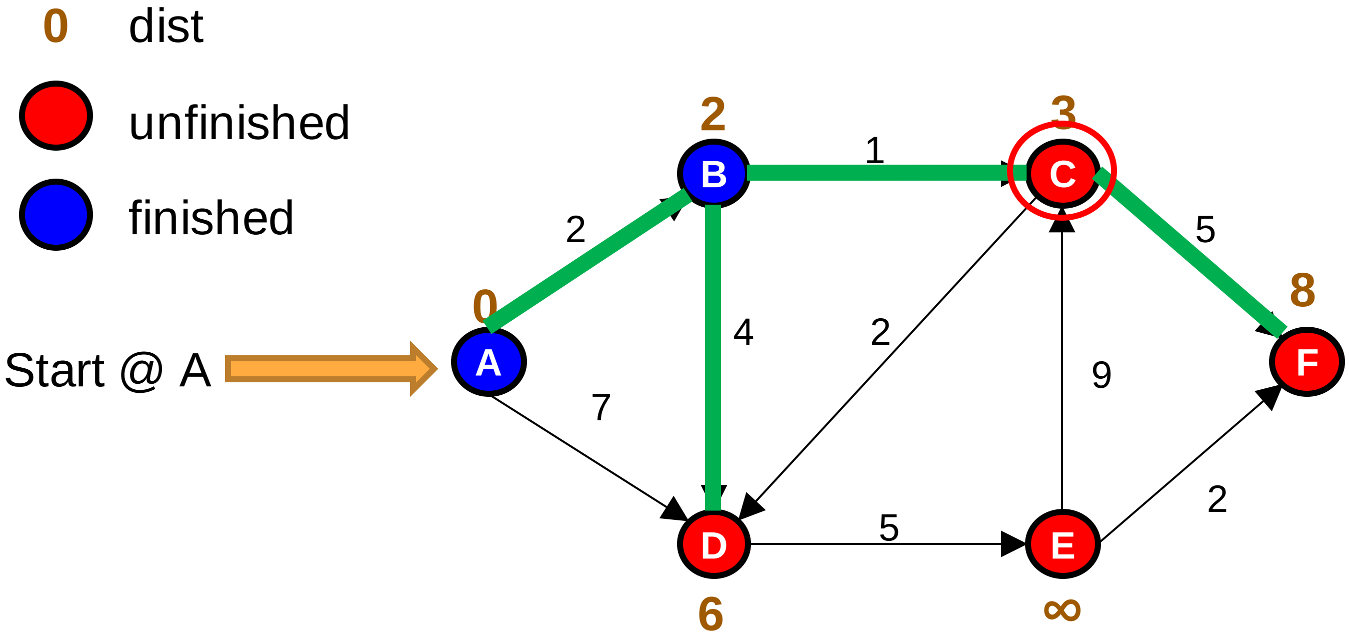


Example of Dijkstra's Algorithm



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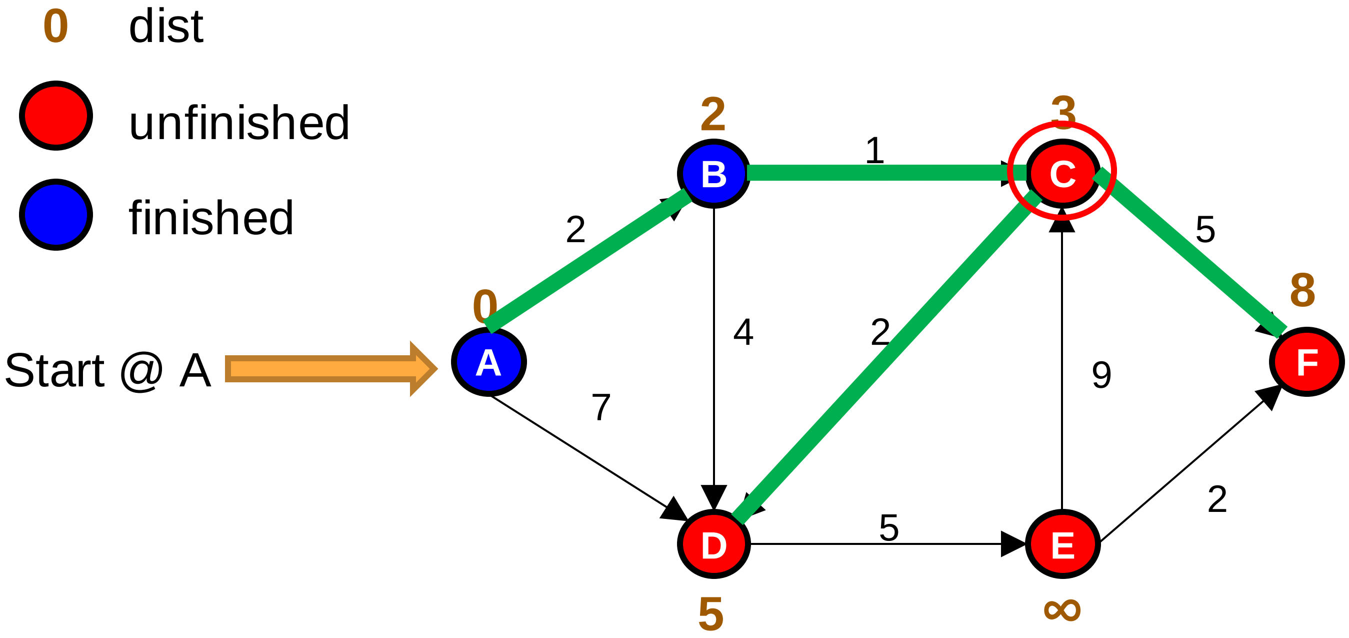

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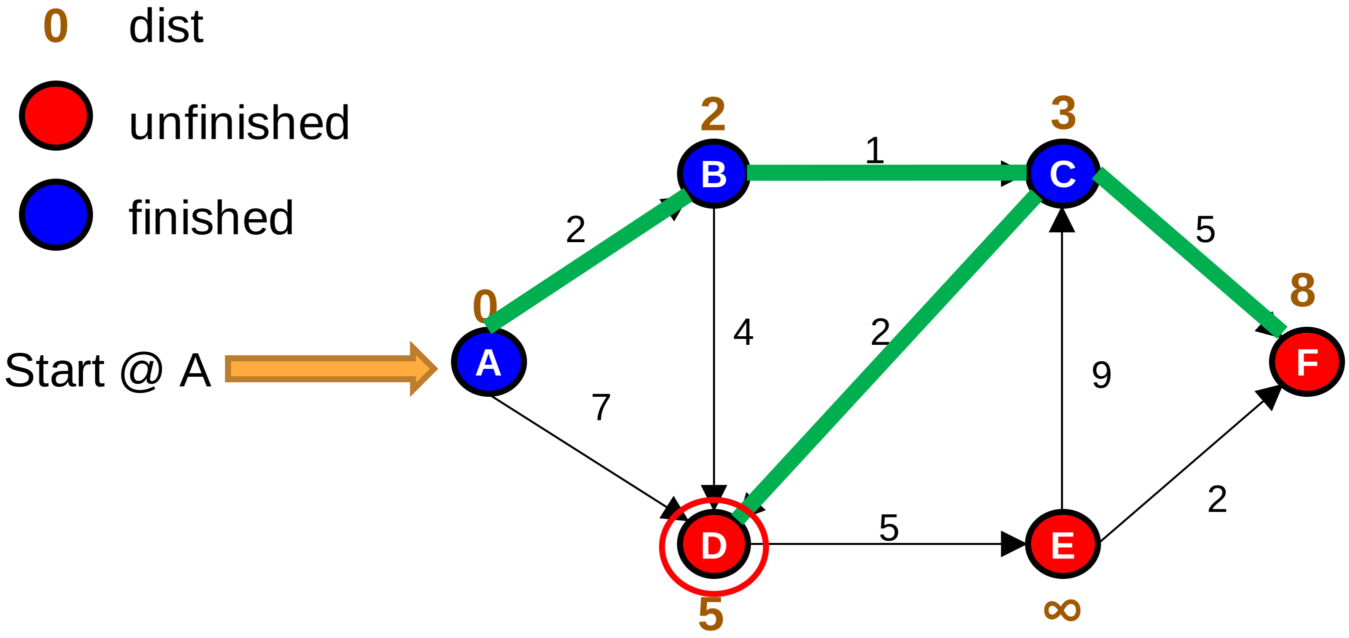

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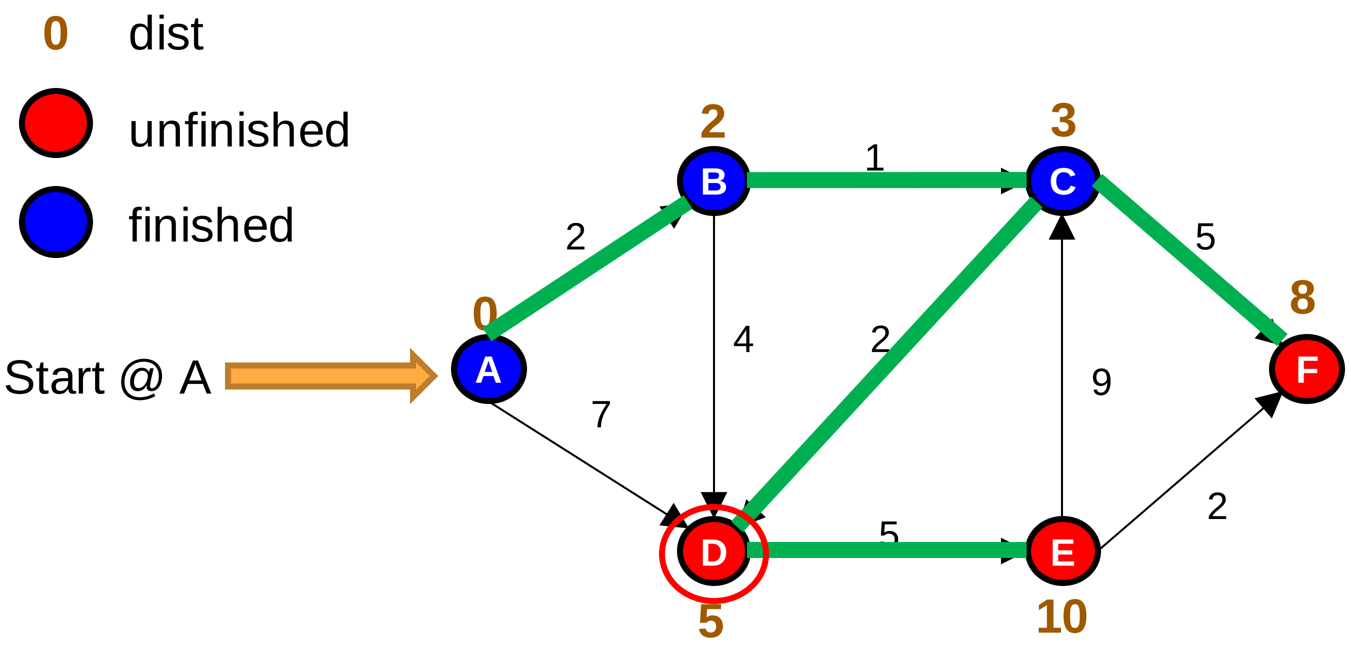

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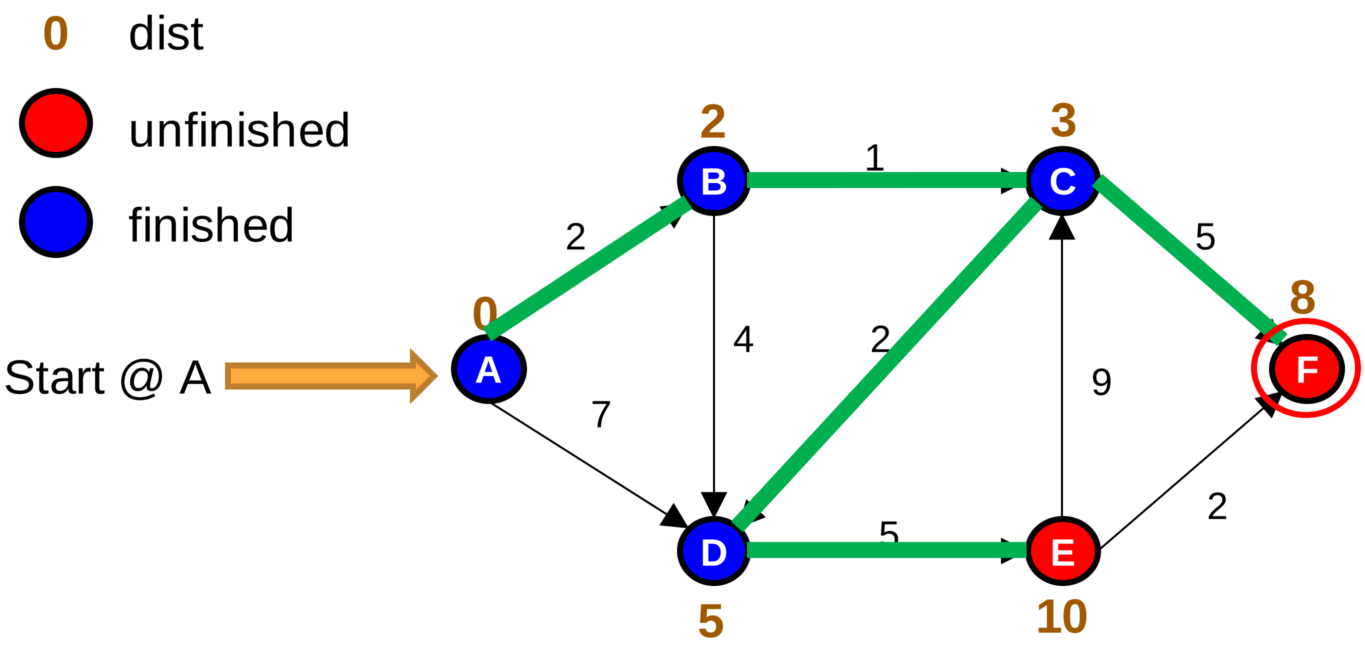

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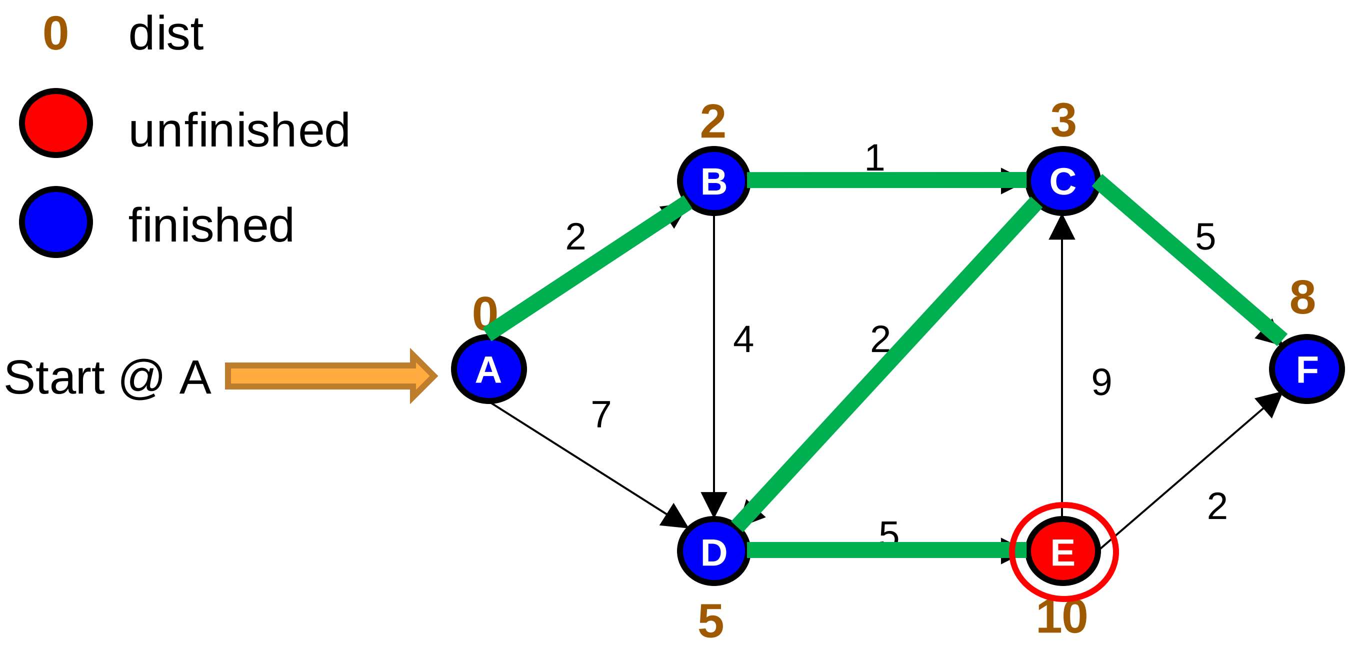
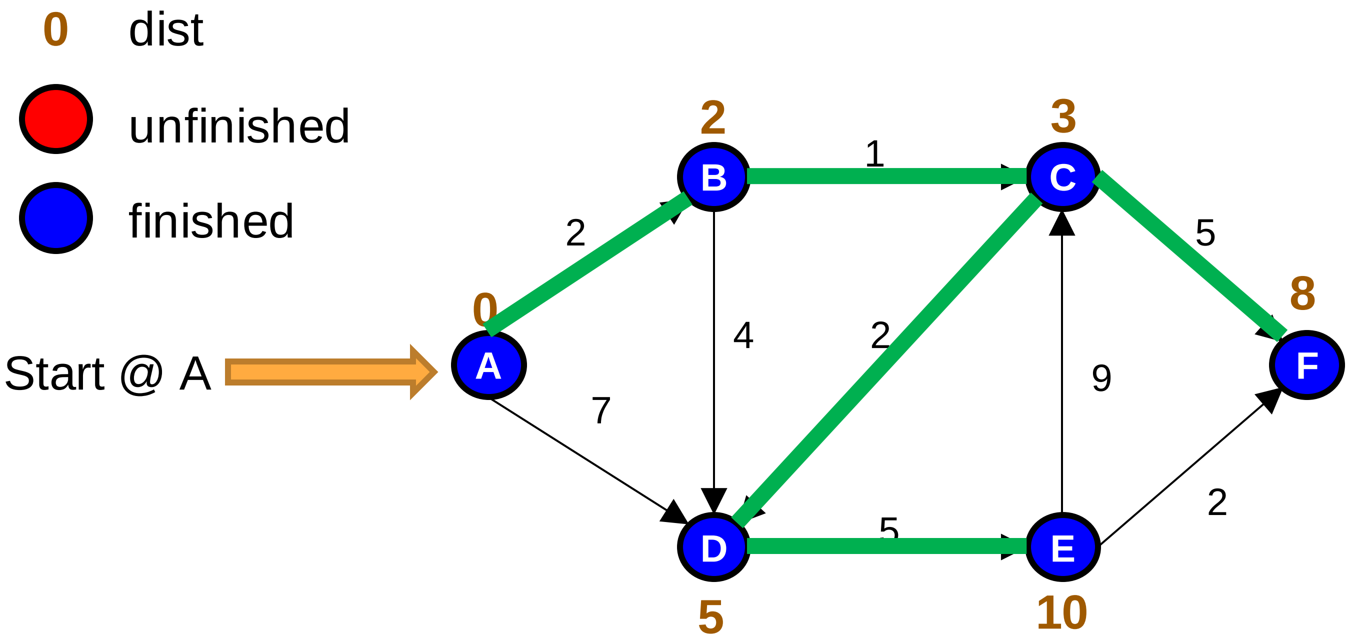


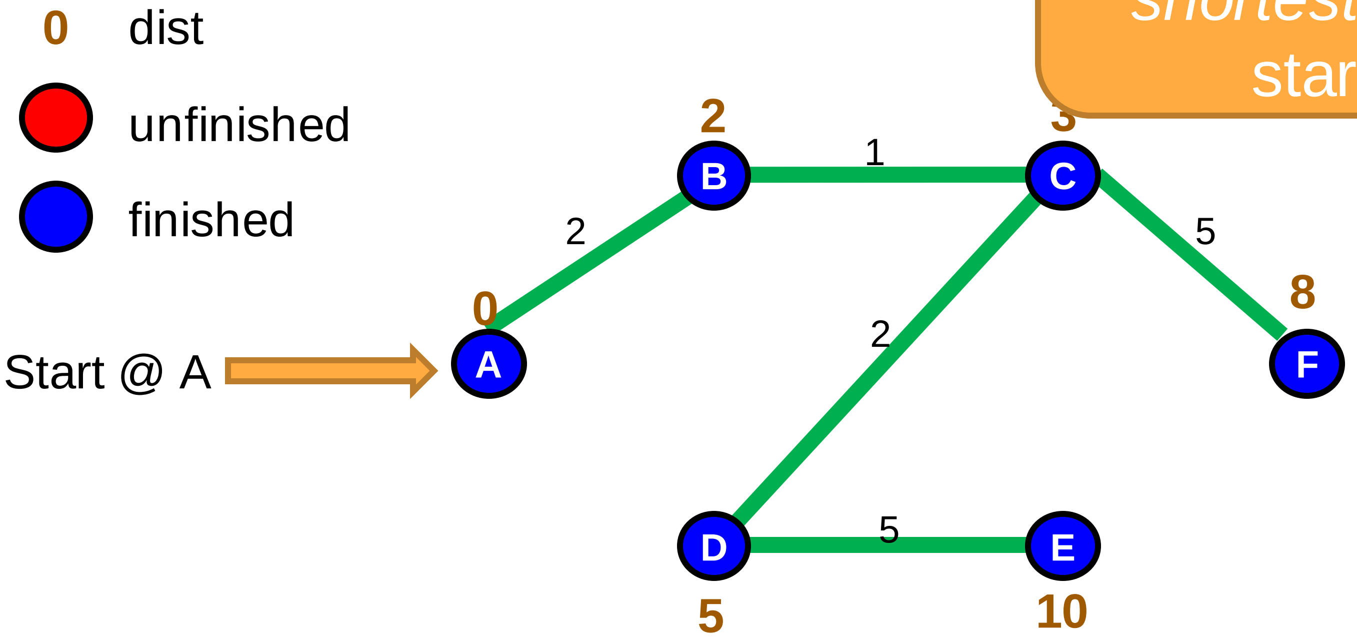
Example of Dijkstra's Algorithm



DONE!

Example of Dijkstra's Algor

As with BFS, parent edges from Dijkstra's algo form a *shortest-path tree* from starting vertex.



Correctness of Dijkstra’s Algorithm

- **Claim**: when we explore the edges out of vertex v, v has its correct shortest-path distance D(start, v) stored in current best estimate v.dist.

- **Pf**: by induction on order of exploration.

- **Bas**: starting vertex is explored first, with its correct shortest-path distance of 0.

Correctness of Dijkstra’s Algorithm

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- **Ind:** suppose the algorithm is about to choose v for exploration.

- Assume that v.dist > D(start, v) (i.e. v’s distance is wrong).

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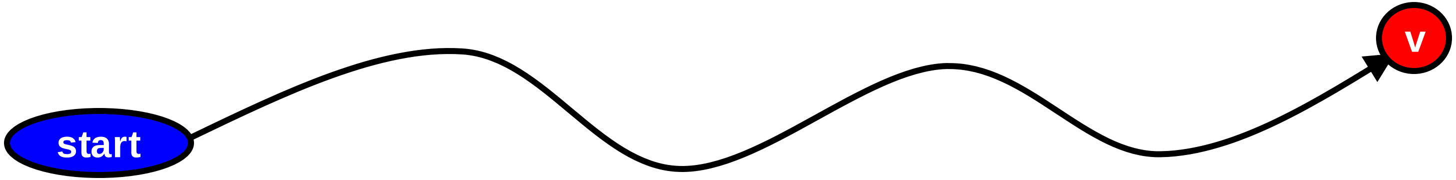
- *[we will derive a contradiction… hence, v.dist must be = D(start, v)]*

Correctness of Dijkstra's Algorithm

- **Ind:** suppose the algorithm is about to choose v for exploration.
- Assume that v.dist > D(start, v) (i.e. v's distance is wrong).
- Consider a *shortest* path from start to v.



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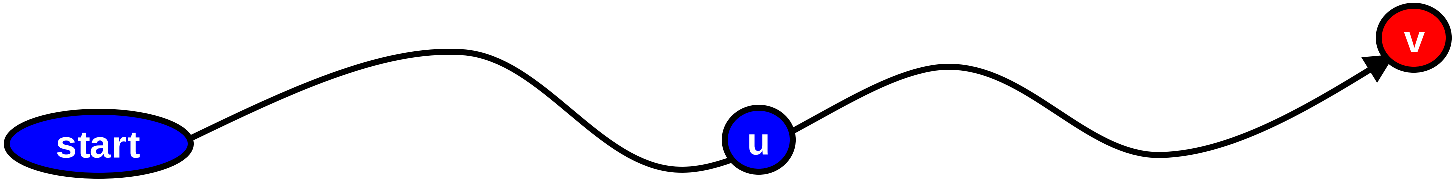


- Let u be last **finished** (i.e., already explored) vertex on this path.

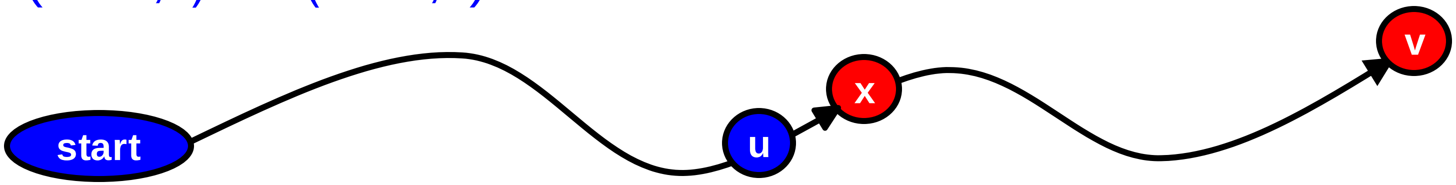
Correctness of Dijkstra's Algorithm

- By IH, u had its correct shortest-path distance when it was explored.
- Moreover, D(start, u) ≤ D(start, v), since *u precedes v* on shortest path to v.
- If edge u → v is on shortest path, then exploring u's outgoing edges assigns v its correct shortest-path distance D(start,v). →←

Correctness of Dijkstra's Algorithm

- If edge u → v is on shortest path, then exploring u's outgoing edges assigns v its correct shortest-path distance D(start,v). →←

- Otherwise, some *other* vertex x lies between u and v on this path, with D(start,x) ≤ D(start,v).



- Since v does *not* have its correct shortest-path distance, **v.dist > x.dist**, and so x would be explored next, not v. →← **QED**

How Do We Track Next Vertex to Explore?

- Maintain collection of unfinished vertices
- At each step, must efficiently **find** vertex v in collection with smallest v.dist and **remove** it
- But vertices' distances may change repeatedly due to relaxation!
- Changes are all in one direction (*decrease*)

What Data Structure Can We Use To Track Distances to Unfinished Vertices?

Use a Priority Queue!

- Maintain priority queue PQ of unfinished vertices, keyed on **dist**
- Initially, every vertex is inserted into PQ w/its starting dist
- At each step, find next vertex to explore by PQ.extractMin()
- Decreasing v.dist is done using v's **`Decreaser`** object
- We assume a map D[ ] from vertices to their Decreasers

Dijkstra's Shortest Path Algorithm w/Prio Queue

- v.dist ← 0; D[v] ← PQ.insert(starting vertex v)
- For all other vertices u
- u.dist ← ∞; D[u] ← PQ.insert(u)
- while (PQ not empty)
- v ← PQ.extractMin()
- for each edge (v,u)
- if (v.dist + w(v, u) < u.dist)
- u.dist ← v.dist + w(v,u)
- D[u].decrease(u)

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Note that Lab 13 creates pairs (vertex, dist) rather than vertices with an internal distance field.

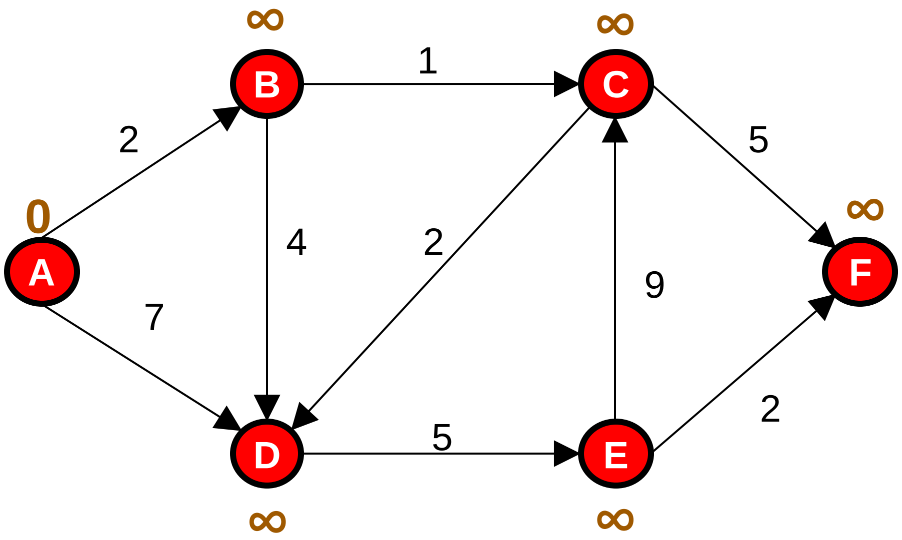
Example of Dijkstra's Algorithm

0 dist

unfinished

finished

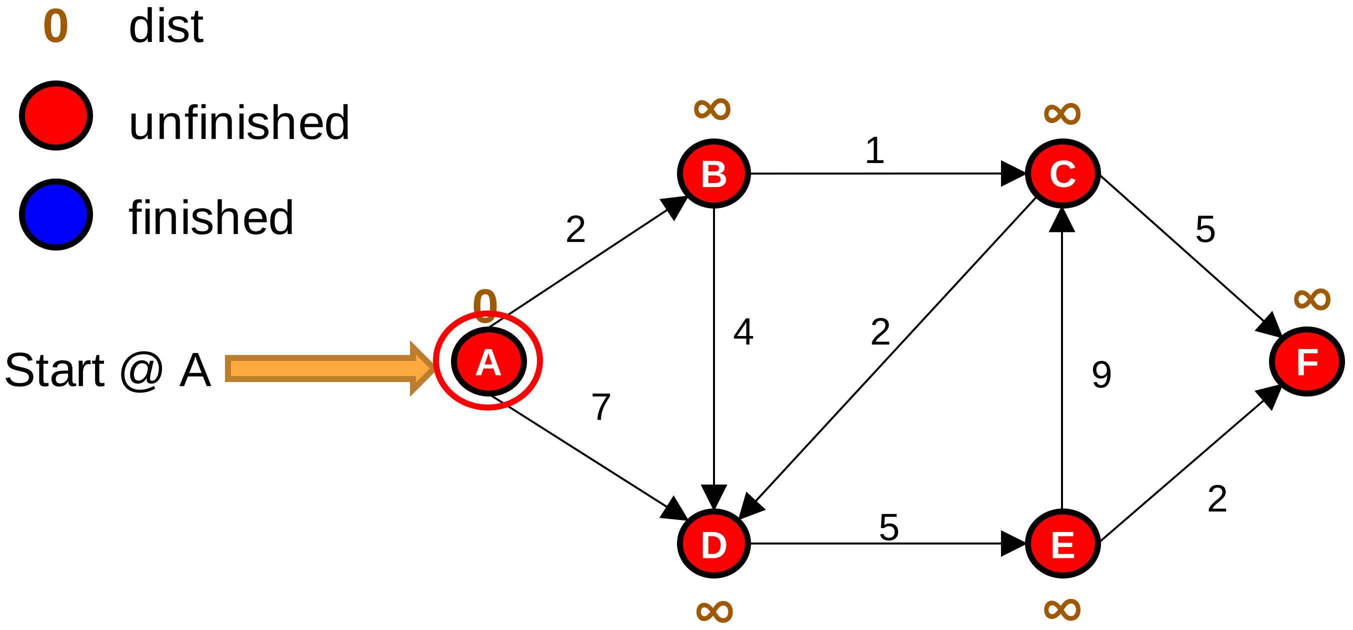
Start @ A



PQ

A	0
B	∞
C	∞
D	∞
E	∞
F	∞

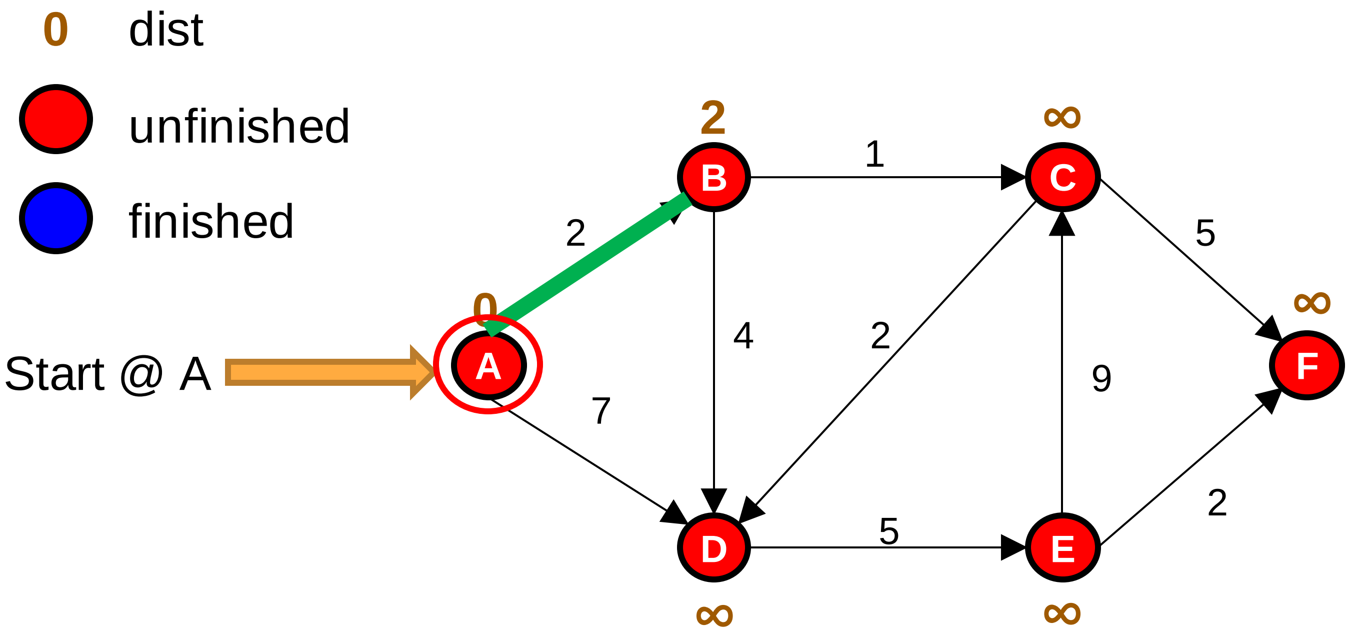
Example of Dijkstra's Algorithm



PQ

A		
	B	∞
	C	∞
	D	∞
	E	∞
	F	∞

Example of Dijkstra's Algorithm



PQ

B	2
C	∞
D	∞
E	∞
F	∞

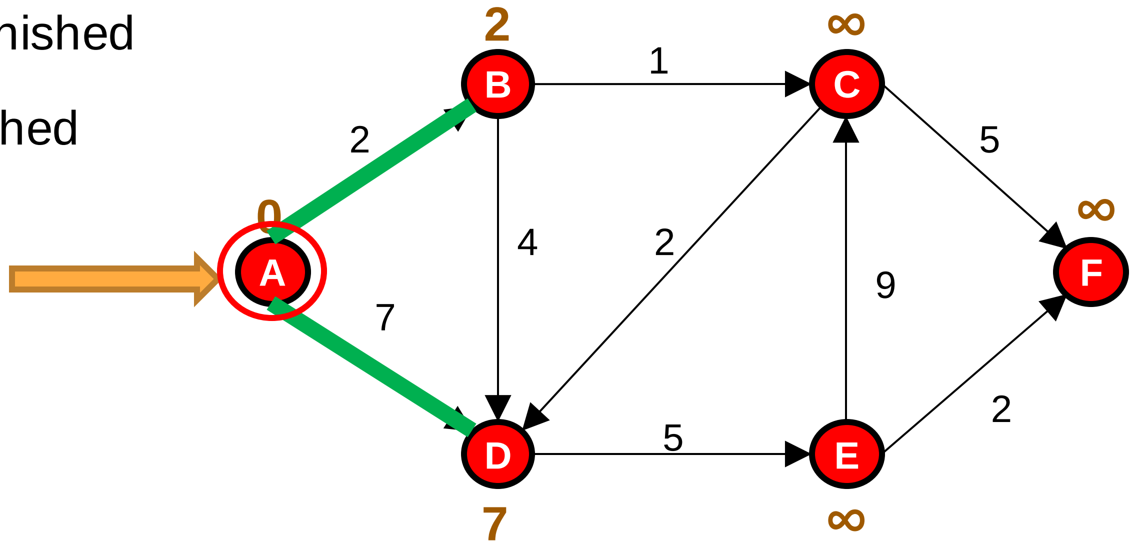
Example of Dijkstra's Algorithm

0 dist

unfinished

finished

Start @ A



PQ

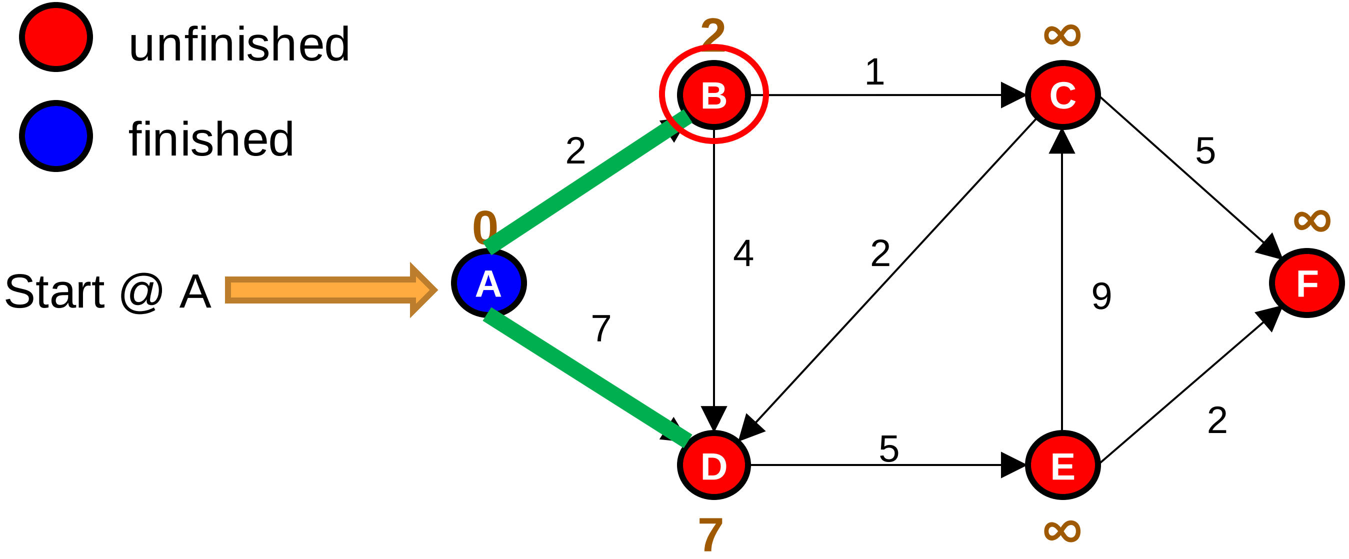
B	2
C	∞
D	7
E	∞
F	∞

Example of Dijkstra's Algorithm

0 dist

unfinished

finished

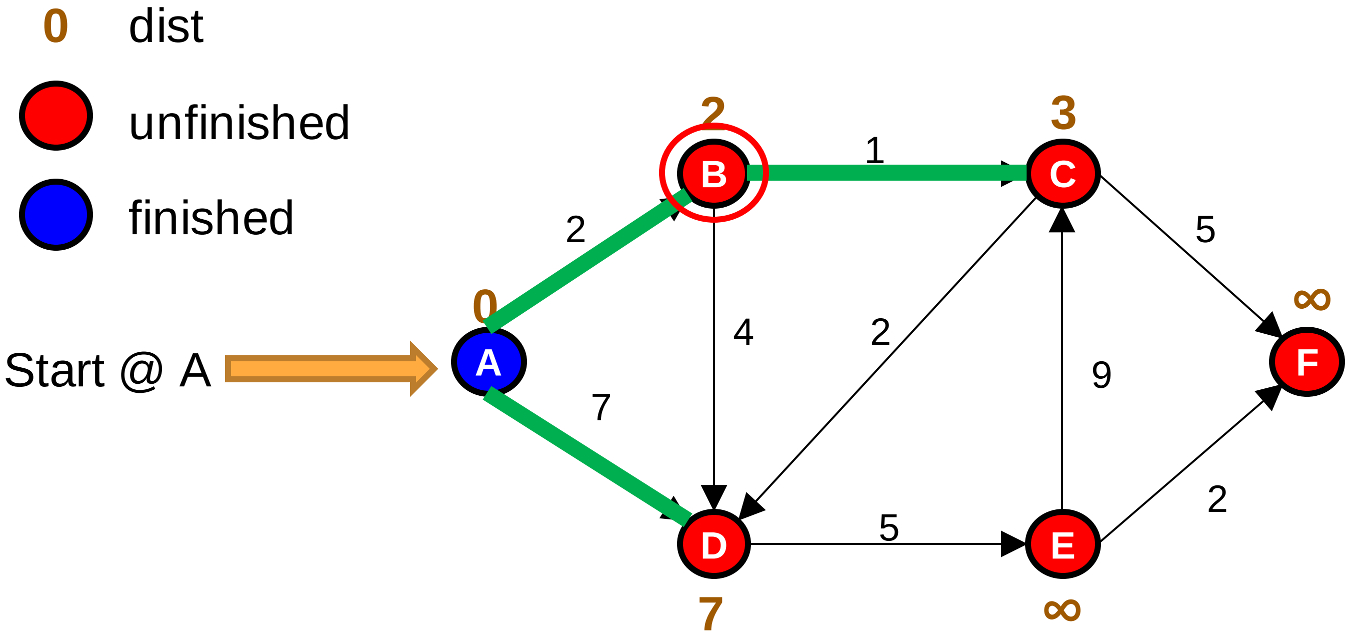


PQ

B

C	∞
D	7
E	∞
F	∞

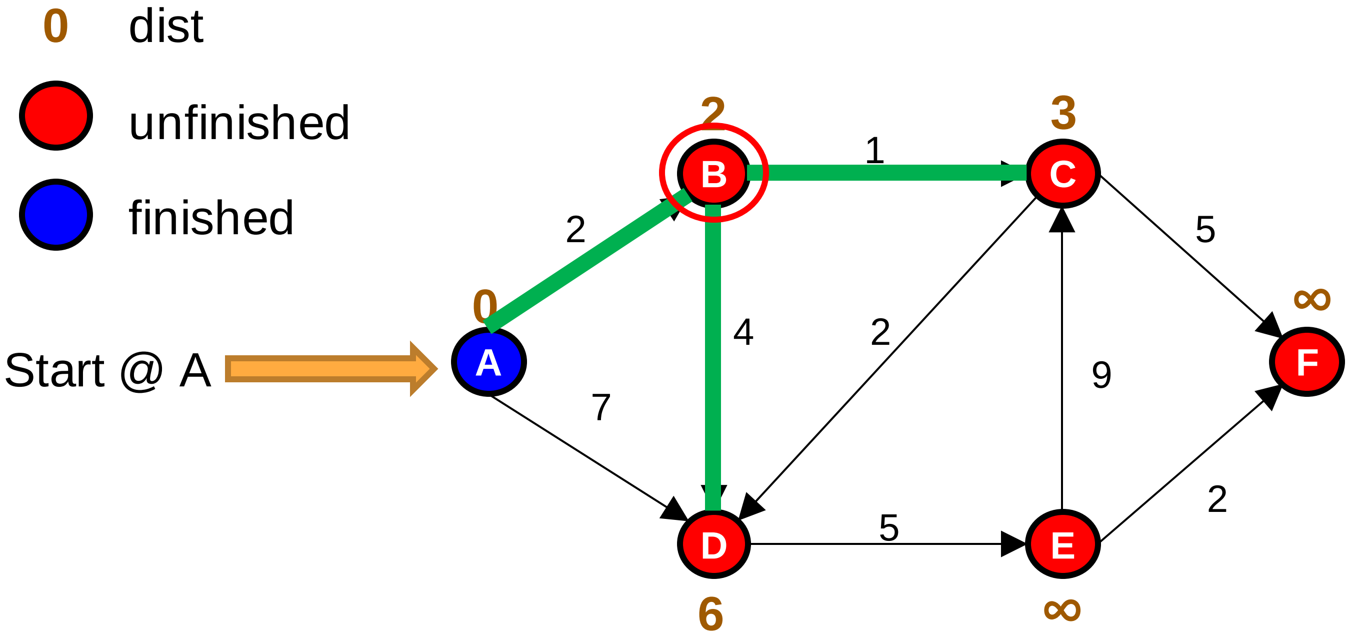
Example of Dijkstra's Algorithm



PQ

C	3
D	7
E	∞
F	∞

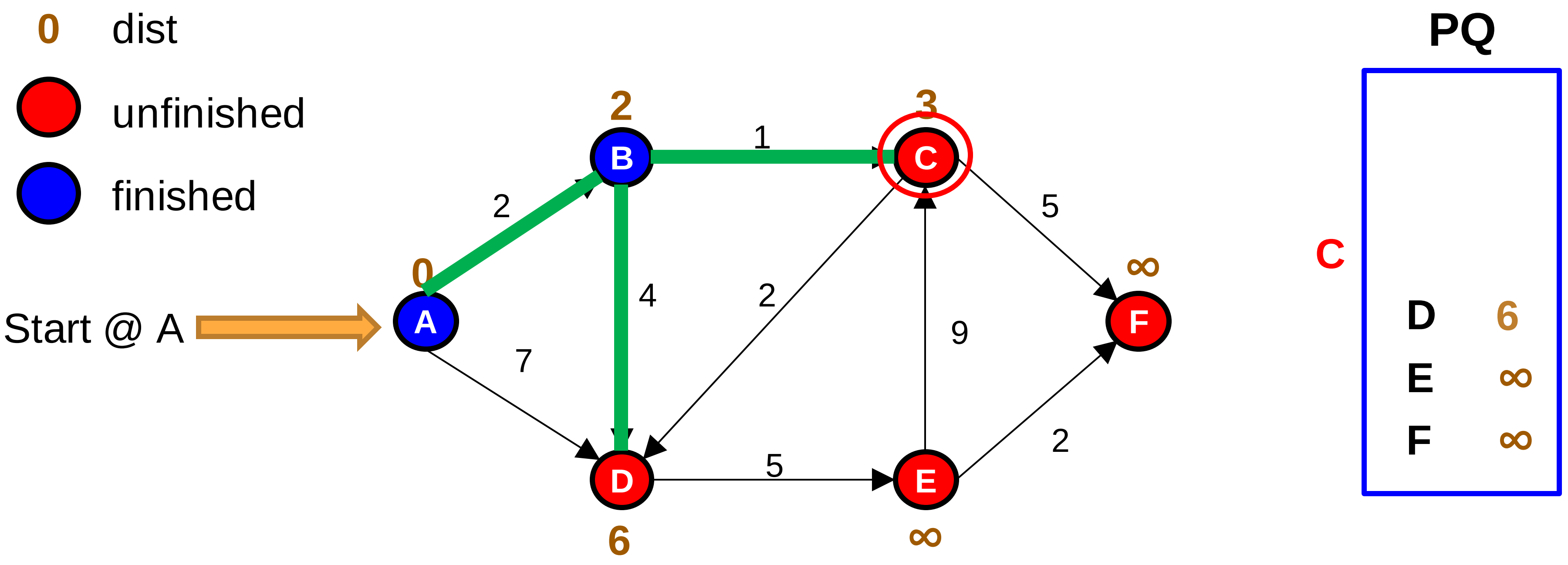
Example of Dijkstra's Algorithm



PQ

C	3
D	6
E	∞
F	∞

Example of Dijkstra's Algorithm

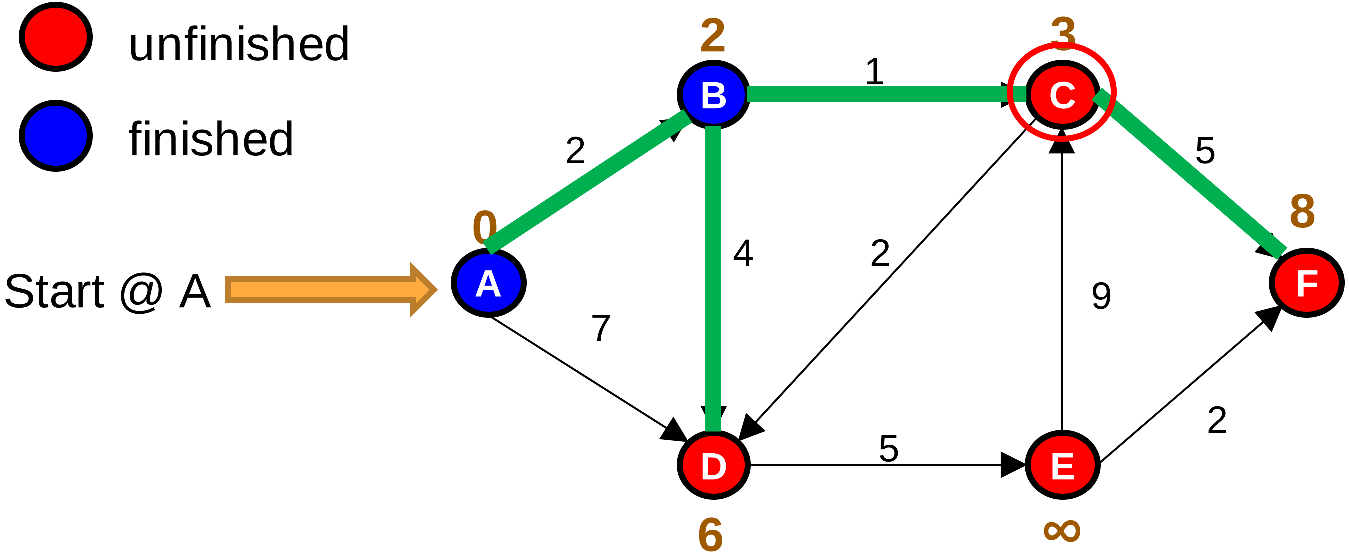


Example of Dijkstra's Algorithm

0 dist

unfinished

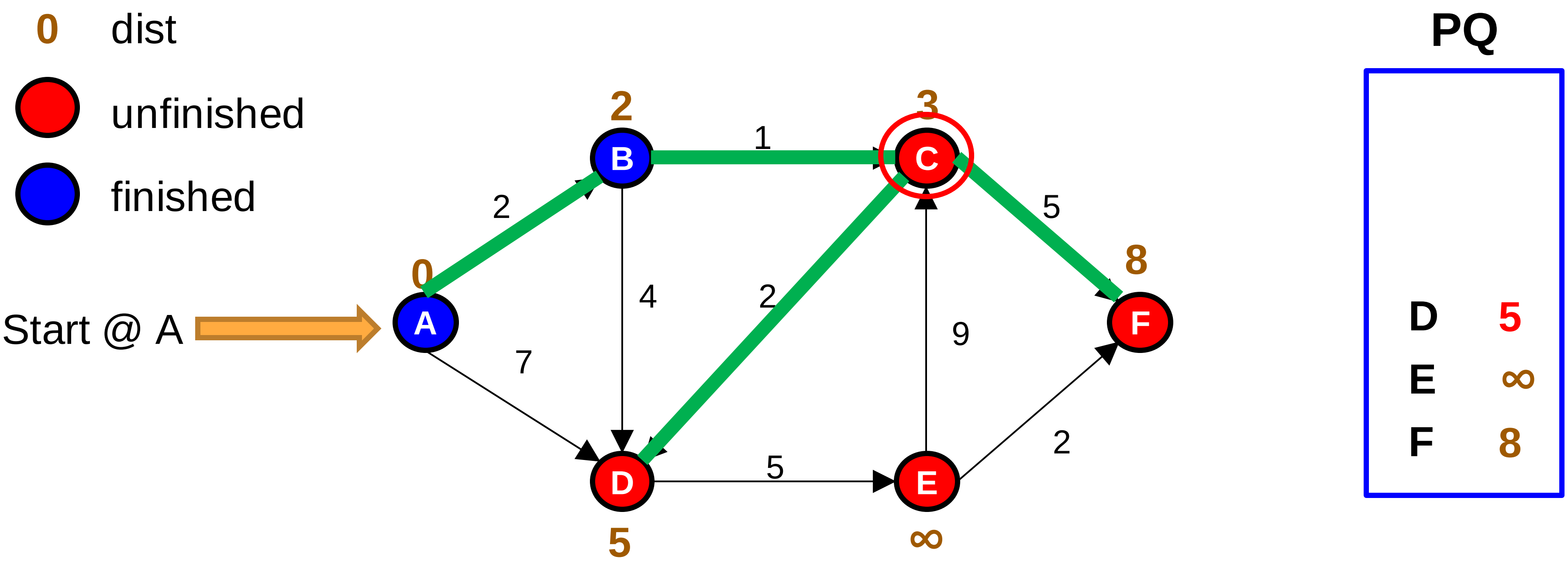
finished



PQ

D	6
E	∞
F	8

Example of Dijkstra’s Algorithm



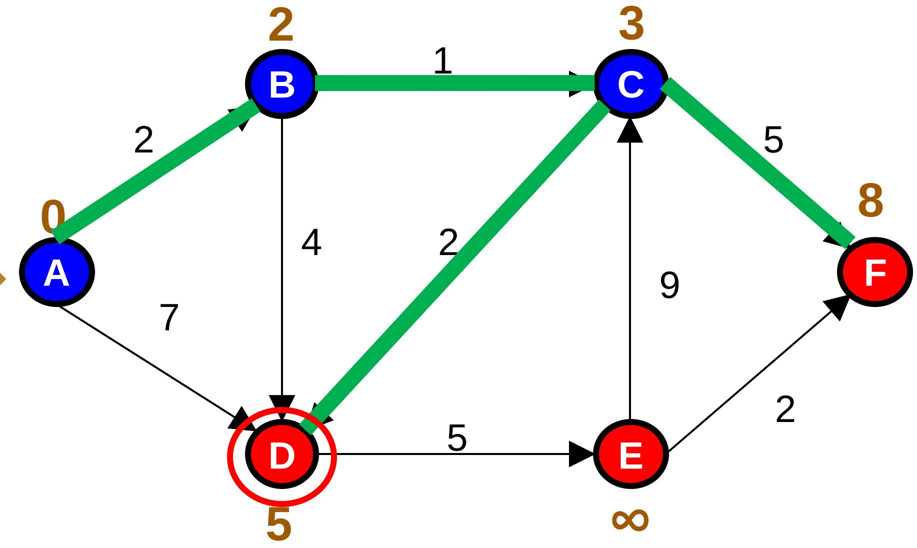
Example of Dijkstra's Algorithm

0 dist

unfinished

finished

Start @ A

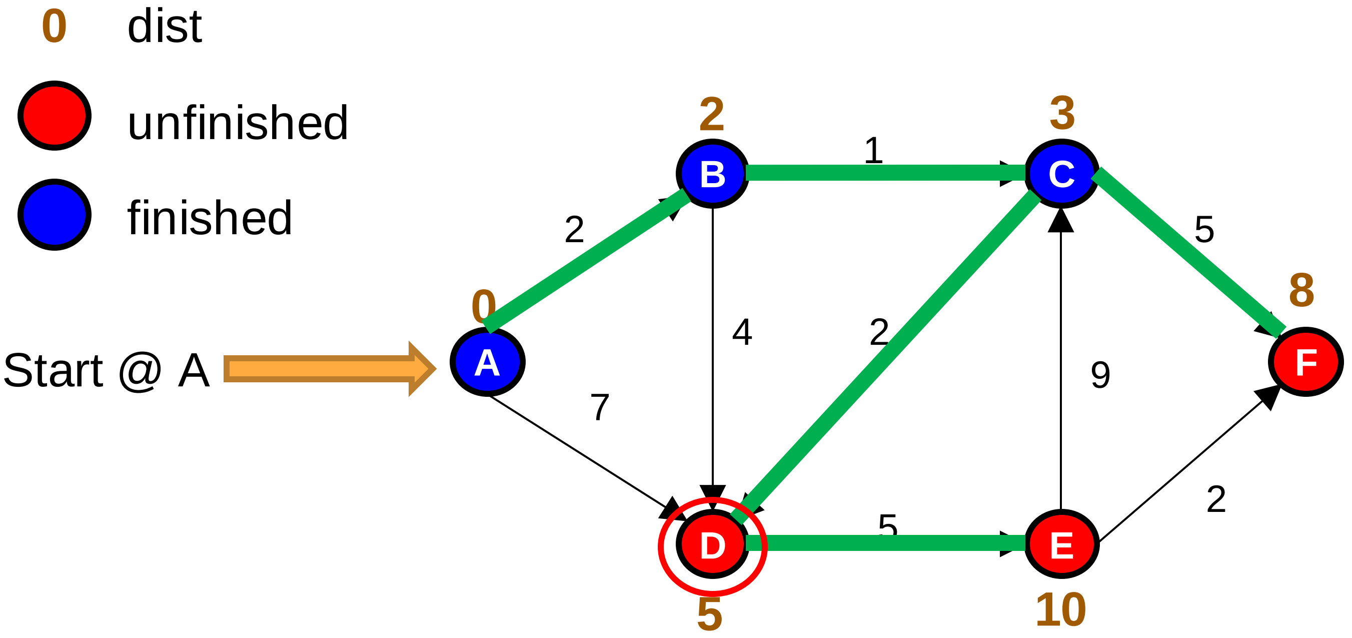


PQ

D

E	∞
F	8

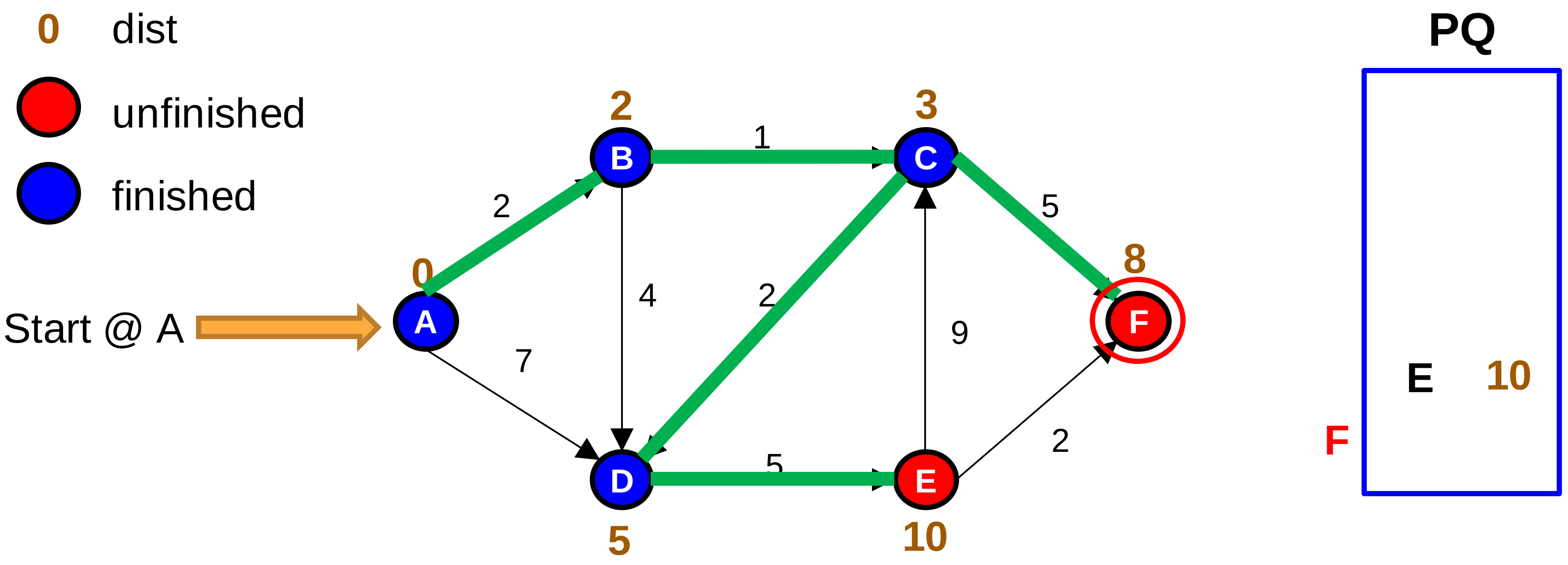
Example of Dijkstra's Algorithm



PQ

E	10
F	8

Example of Dijkstra's Algorithm



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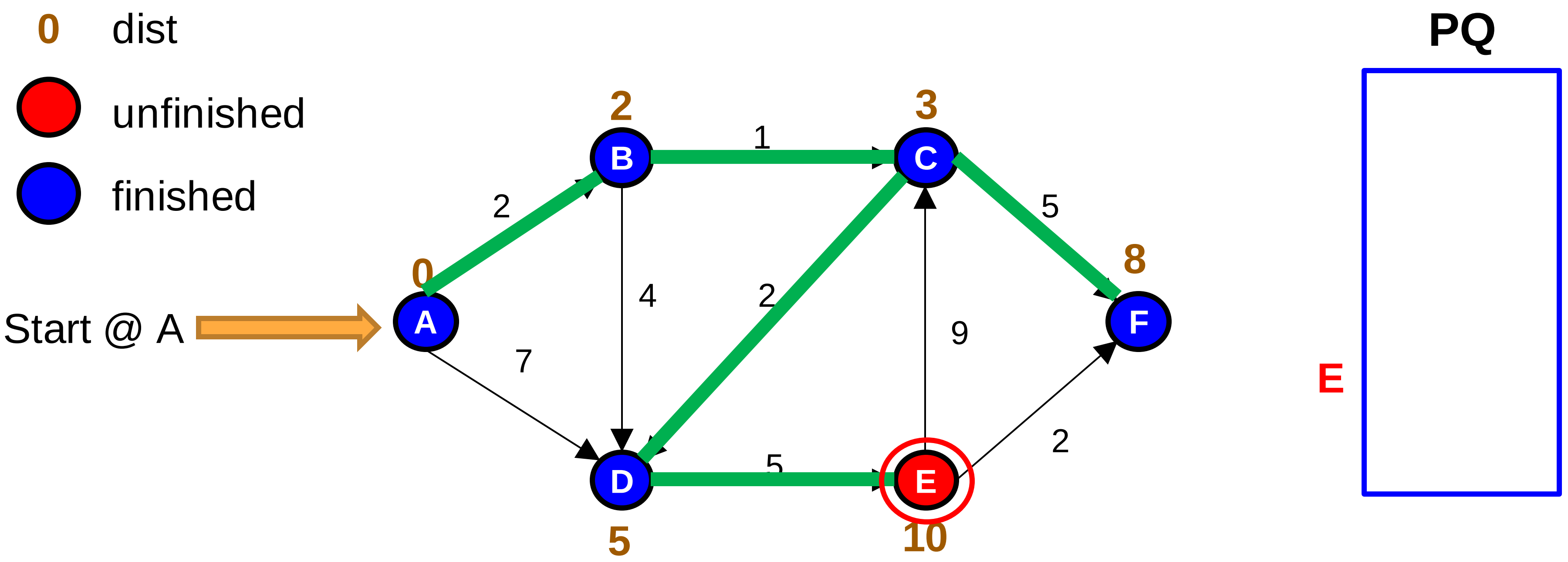

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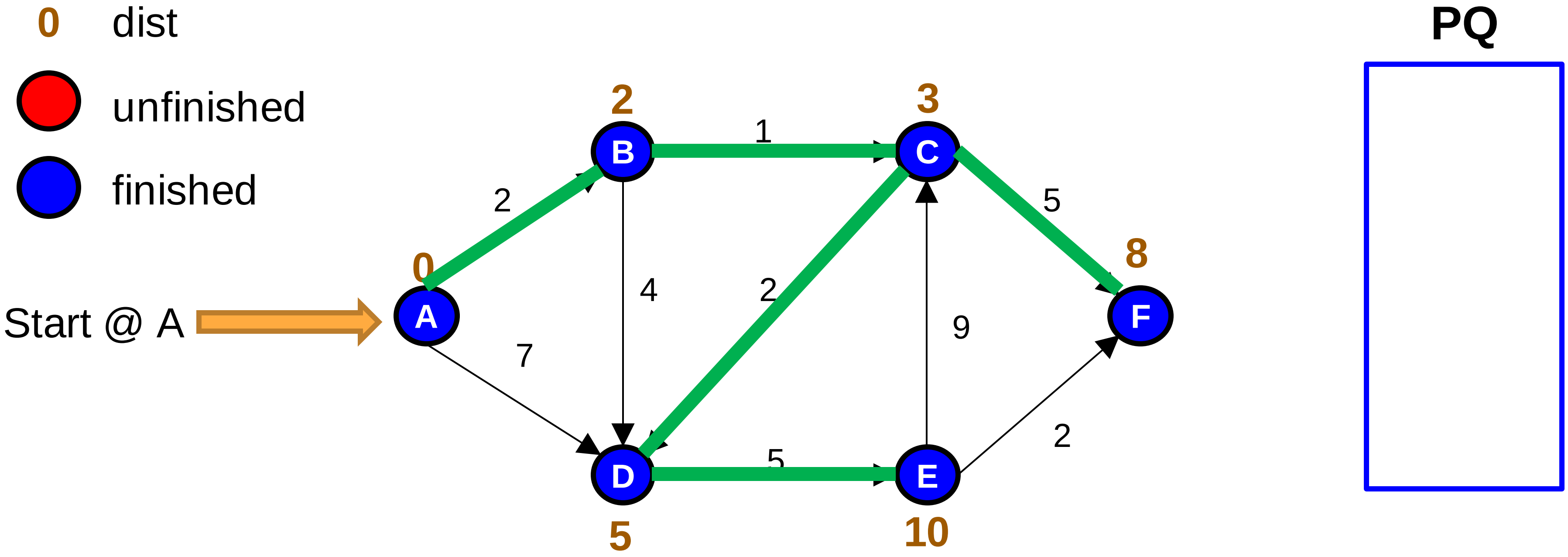

DONE!

Running Time of Dijkstra's Algorithm

- For each **vertex**, we do one PQ insert() and one PQ extractMin()

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- Times for PQ operations using binary heap are all **???**

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- Hence, total cost is $|V|(T_{insert} + T_{extractMin}) + |E|\ T_{decrease}$
- Times for PQ operations using binary heap are all **$\Theta(\log |V|)$**
- **Hence, algorithm runs in time $\Theta((|V| + |E|) \log |V|)$**

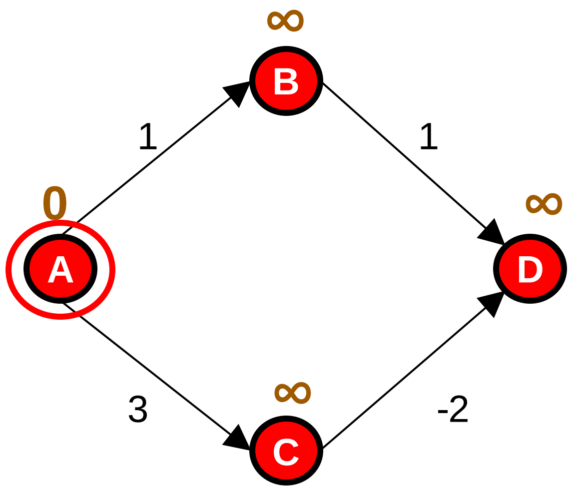
Running Time of Dijkstra's Algorithm

- For each **vert** ... O extractMin()
- For each **ed**
- Hence, total ... se
- Times for PC ... **(log |V|)**
- **Hence, algorithm runs in time Θ((|V| + |E|) log |V|)**

Slightly faster algorithms are possible for dense graphs, using much fancier heaps with O(1) decrease time.

Why Must Edge Weights Be Non-Negative?

- With negative weights, Dijkstra's algorithm does not necessarily find a shortest (smallest total sum of edge weights) path.



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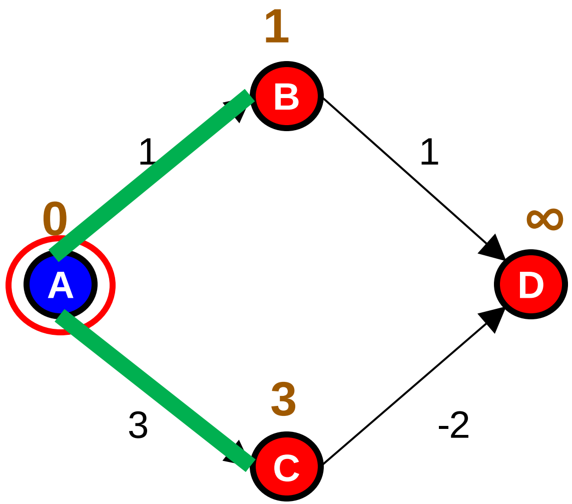

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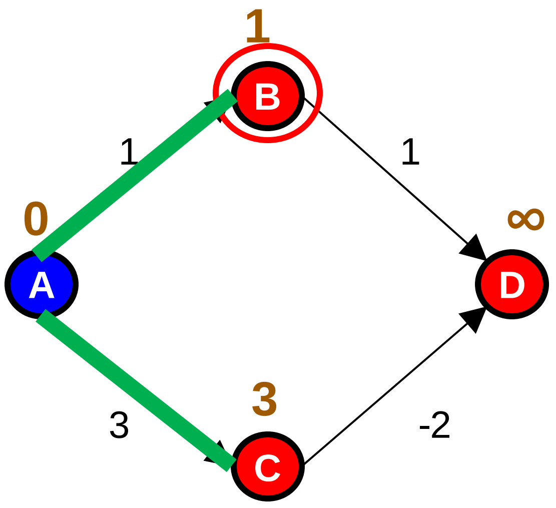

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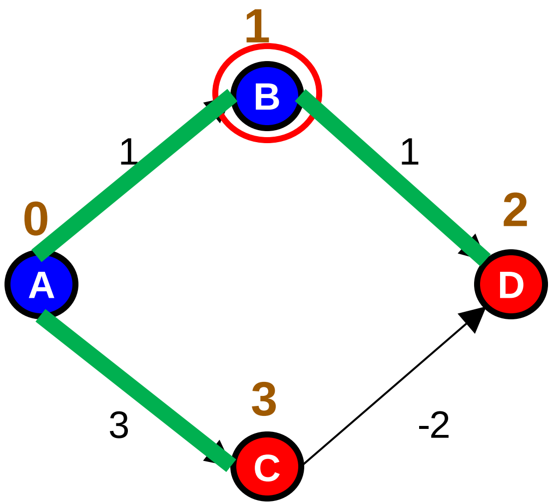

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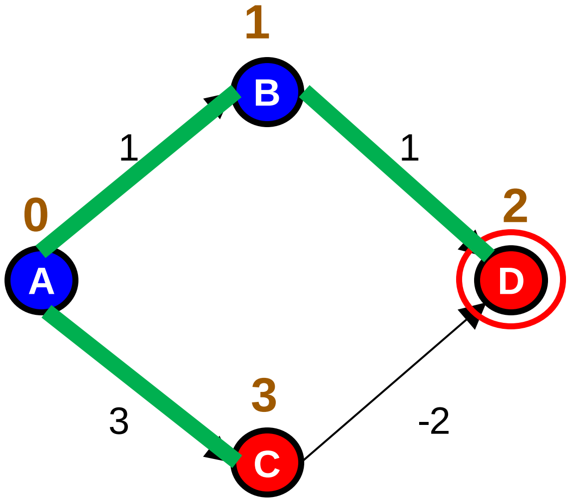

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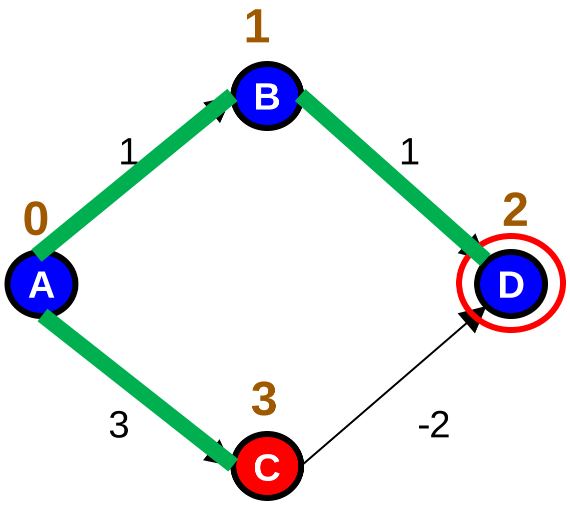

D is *finalized* with distance 2.

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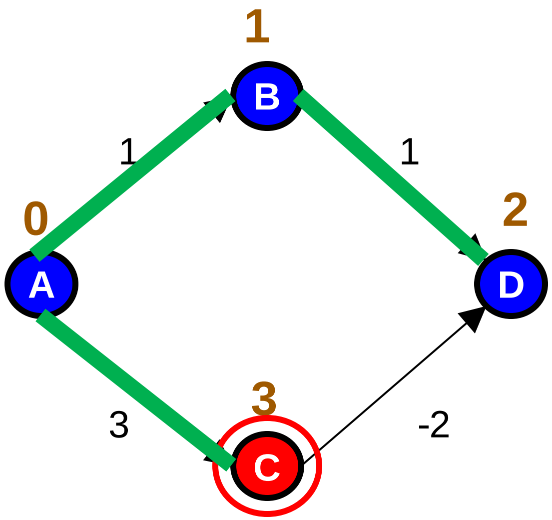

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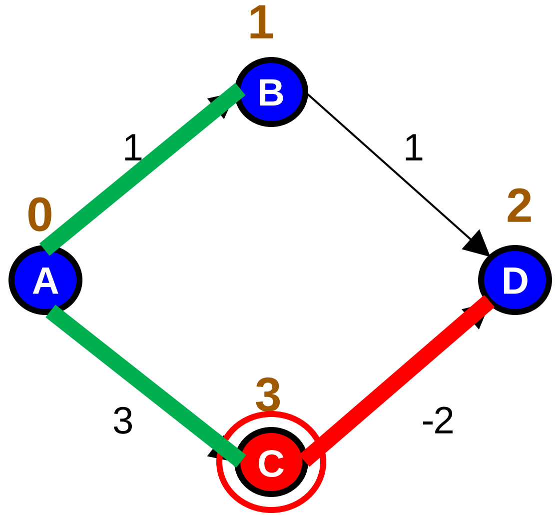

But *shortest* path is of length 3 + -2 = 1

Alternatives to Dijkstra

- If negative-weight edges are allowed…
- May use *Bellman-Ford* algorithm ($\Theta(|V||E|)$)
- If shortest-path distances are desired between every pair of vertices…
- May use *Floyd-Warshall* algorithm ($\Theta(|V|^3)$)
- (*Other approaches may be better for sparse graphs*)