Examination Number 1

Answer all questions. Show all work.

1.(20 points)
V = \{(x_1, x_2, x_3) | x_1 + 2x_2 + x_3 = 0\} is a vector space.
(a) What is the dimension of V?
(b) Find a basis for V.
(c) Find an orthogonal basis for V.
(d) Find the projection of \( w = (1, 1, 1) \) on V. (This is a vector).

2.(15 points)
l is the vector \((-i + 5j + 3k)\)
Find the equation of the plane through \( P_0(1, 0, 0) \) whose normal is parallel to l.

3.(20 points)

\[
A = \begin{pmatrix}
2 & 0 & 1 \\
0 & 1 & 3 \\
0 & 3 & 1
\end{pmatrix}
\]
(a) Find all eigenvalues and corresponding eigenvectors of A.
(b) Compute \((A - 2I)(A^2 - 2A - 8I) + A\). (Hint: do not try to do it directly)

4.(20 points)
V = \{(x_1, x_2, x_3) | -x_1 + 2x_2 + 3x_3 = 0, -4x_1 + x_2 + x_3 = 0\}
(a) Is V a vector space?
If so
(b) What is the dimension of V?
(c) Find a basis for V.

5.(25 points)

\[
A = \begin{pmatrix}
2 & -1 & 0 \\
0 & -2 & 1 \\
1 & 0 & 1
\end{pmatrix}
\]
Compute \text{trace}(A), \text{determinant}(A), A^{-1}, \text{the characteristic polynomial of} \ A, \text{the rank of} \ A, \text{the null space of} \ A.
\[ \mathbf{V} = \left\{ (x_1, x_2, x_3) \mid x_1 + 2x_2 + x_3 = 0 \right\} \text{ is a subspace.} \]

1. What is the dimension of \( \mathbf{V} \)?
2. Find a basis for \( \mathbf{V} \).
3. Find an orthogonal basis for \( \mathbf{V} \).
4. Find the projection of \( \mathbf{w} = (1, 2, 4) \) onto \( \mathbf{V} \).

\[ x_3 = -x_1 - 2x_2 \]

For any vector \( \mathbf{x} \in \mathbf{V} \),

\[ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} x_1 + \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} x_2 \]

So \( \mathbf{v}_1, \mathbf{v}_2 \) span \( \mathbf{V} \).

\( \mathbf{v}_1, \mathbf{v}_2 \) are linearly independent vectors:

\[ c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = \mathbf{0} \Rightarrow c_1 = 0 \quad \text{and} \quad c_2 = 0 \]

Thus, \( \mathbf{v}_1, \mathbf{v}_2 \) is a basis for \( \mathbf{V} \).

Dimension of \( \mathbf{V} = 2 \).

To find an orthogonal basis for \( \mathbf{V} \), set \( \mathbf{w} = \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \).

\[ \mathbf{v}_1 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} - \frac{2}{9} \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} - \frac{2}{9} \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 7/9 \\ 4/9 \\ -10/9 \end{pmatrix} \]

Check:

\[ \mathbf{v}_1 \cdot \mathbf{v}_1 = \begin{pmatrix} 7/9 \\ 4/9 \\ -10/9 \end{pmatrix} \cdot \begin{pmatrix} 7/9 \\ 4/9 \\ -10/9 \end{pmatrix} = 1 \]

\[ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \]

Powers of \( \mathbf{v}_1 \):

\[ \mathbf{v}_1^2 = -\frac{1}{3} \mathbf{v}_1 = -\frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ -4/3 \end{pmatrix} \]

Powers of \( \mathbf{v}_2 \):

\[ \mathbf{v}_2^2 = \frac{4}{9} \mathbf{v}_2 = \frac{4}{9} \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 4/9 \\ 8/9 \\ -16/9 \end{pmatrix} \]

\[ \text{proj}_\mathbf{V} \mathbf{w} = 0 \mathbf{v}_1 - \frac{1}{3} \mathbf{v}_2 = -\frac{1}{3} \begin{pmatrix} 7/9 \\ 4/9 \\ -10/9 \end{pmatrix} = \begin{pmatrix} -7/27 \\ -4/27 \\ 10/27 \end{pmatrix} \]
\( \vec{r} = \text{position vector of a point on the plane } P \)

\[(\vec{r} - \vec{r}_0) \cdot \vec{l} = 0 \]

\[\vec{r} \cdot \vec{l} = \vec{r}_0 \cdot \vec{l} \]

\[\vec{r} = (x, y, z), \quad \vec{r}_0 = (1, 0, 0), \quad \vec{l} = (-1, 5, 3) \]

\[-x + 5y + 3z = -1 \]
(a) $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}$

$|3I - A| = \begin{vmatrix} 3-2 & 0 & -1 \\ 0 & 3-1 & -3 \\ 0 & -3 & 3-1 \end{vmatrix} = 0$

$(\lambda - 2) \cdot (\lambda - 1)^2 - 9 = 0$

$(\lambda - 2) \cdot (\lambda^2 - 2\lambda - 9) = 0$

$\lambda = 2, \lambda = 1 \pm 3$

$x_1 = 2$

$x_2 = 1$

$x_3 = -3$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$

(b) $(A - 2I) (A^2 - 8A - 9I) + A = A$

By Cayley-Hamilton theorem.
(4) $V = \{ x \in \mathbb{R}^3 \mid -x_1 + 2x_2 + 3x_3 = 0, -x_1 + x_2 - x_3 = 0 \}$

(a) $\lambda \in \sigma(V)$

(b) What is the dimension of $V$?

(c) Find a basis for $V$.

(5) The operation $\oplus$ defines a vector space property for $V$, and $V$ is a subspace of $\mathbb{R}^3$.

\[ y = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} x \]

\[ A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \]

(5) Compute $\det A$, $A^{-1}$, characteristic polynomial of $A$.

(a) $\det A = 2 \cdot 2 - 1 = 3$

(b) $\det A = 2 \cdot 2 - 1 = 3$

(c) $P_A(x) = \det (A - xI) = \begin{vmatrix} 2-x & -1 & 0 \\ 0 & 2-x & -1 \\ 1 & 0 & 1-x \end{vmatrix} = -(4-x^2)(1-x) - (-4-x^3+7x) = -(4-x^2)(1-x) - (-4-x^3+7x) = (4-x^2)(1-x) + 1$

\[ P_A(x) = \begin{vmatrix} 2-x & -1 & 0 \\ 0 & 2-x & -1 \\ 1 & 0 & 1-x \end{vmatrix} = -(4-x^2)(1-x) - (-4-x^3+7x) = (4-x^2)(1-x) + 1 \]

[Additional calculations and explanations for (6)]
(5) \text{if } A \leq 10 \Rightarrow \text{number is even} \quad \text{so multiply } A \times 2 \Rightarrow \text{new } A = 3

(6) \text{if } A > 10 \Rightarrow A \times 2 \text{ is even, so} \quad A = 3 \Rightarrow \text{new } A = 0

As result, \( A = 0 \).