This exam contains nine multiple-choice problems worth two points each, eight true-false problems worth one point each, eight short-answer problems worth one point each, and one free-response problem worth six points, for an exam total of 40 points.

Part I. Multiple-Choice (two points each)

Clearly fill in the oval on your answer card which corresponds to the only correct response.

1. Miss Kitti has 22 children in her first-grade classroom, and she assigns jobs to seven children each week: one line leader, one fish feeder, one plant waterer, one eraser cleaner, two paper passers, and one paper collector. In how many ways can these jobs be assigned?

(A) 5,040
(B) 74,613
(C) 170,544
(D) 341,088
(E) 7,162,848
(F) 53,721,360
(G) 429,770,880
(H) 456,631,560
(I) 859,541,760
(J) 6,446,563,200

2. Seven dice are rolled. What is the probability that the result will be six odd numbers and one even number?

(A) .0014
(B) .0078
(C) .0097
(D) .0180
(E) .0182
(F) .0428
(G) .0547
(H) .1094
(I) .1301
(J) .1429
3. Exactly one male tribute and one female tribute are to be randomly selected from the youth of District 12 to participate in the 74th annual Hunger Games. Gale's name is entered 42 times to be the male tribute, and Katniss is entered 20 times to be the female tribute. If there are 2500 entries for the male tribute and 2400 entries for the female tribute, what is the probability that either Gale's name or Katniss's name will be drawn, but not both? (Note that the names of the males are in one large glass ball and the names of the females are in another; they are not mixed.)

(A) .00014
(B) .01237
(C) .01251
(D) .01265
(E) .02485

4. Suppose 44% of college students like jazz, 54% like hip hop, and 80% like one or the other or both types of music. If a college student likes hip hop, what is the probability that he or she also likes jazz?

(A) .3333
(B) .3704
(C) .4091
(D) .4545
(E) .5000
(F) .5556
(G) .6923
(H) .7692
(I) .7222
(J) .8148

\[
P[A] = .44 \quad P[B] = .54 \quad P[A \cup B] = .8
\]

\[
P[A \cap B] = .44 + .54 - .8 = .18
\]

\[
P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{.18}{.54} = .3333
\]
5. When a message is sent within a certain spy network, the person receiving it must be sure that it is authentic. This authentication is accomplished using a secret enciphering key. Unfortunately, the secret key could be discovered by an enemy and then used to make an unauthentic message appear to be authentic. Suppose that 95% of all messages are authentic. Suppose further that 3% of unauthentic messages are sent using the correct key and that all authentic messages are sent using the correct key. Find the probability that a message sent with the correct key is, in fact, authentic.

\[ P[A] = 0.95 \quad P[B|A'] = 0.03 \quad P[B|A] = 1 \]

Use Bayes’ Theorem,

\[ P[A|B] = \frac{(1)(0.95)}{(1)(0.95) + (0.03)(0.05)} = 0.99842 \]

6. Let \( f(x) \) be a density for a discrete random variable \( X \). What is the correct computational formula for \( E[X^3] \)?

(A) \[ \sum_x x^3 \]

(B) \[ \sum_x x^3 f(x) \]

(C) \[ \sum_x x f(x^2) \]

(D) \[ \sum_x x(f(x))^3 \]

(E) \[ \sum_x x^3 f(x^3) \]

(F) \[ \sum_x x^3(f(x))^3 \]

(G) \[ \left[ \sum_x x \right]^3 \]

(H) \[ \left[ \sum_x x f(x) \right]^3 \]
7. Patrice is excellent at jumping rope. She misses, on average, only one in 40 jumps. If she begins now to jump rope, find the probability that her first miss will be somewhere between her 30th and her 50th jumps (including 30 and 50).

\[ \text{geometric} \quad p = \frac{1}{40} \]

\[ P[30 \leq X \leq 50] = F(50) - F(29) \]

\[ = \left( 1 - \left( \frac{39}{40} \right)^{50} \right) - \left( 1 - \left( \frac{39}{40} \right)^{29} \right) = \left( \frac{39}{40} \right)^{29} - \left( \frac{39}{40} \right)^{50} \]

\[ = .1979 \]

\[ \text{or} \]

\[ \text{geometriccdf} \left( \frac{1}{40}, 50 \right) - \text{geometriccdf} \left( \frac{1}{40}, 29 \right) = .1979 \]

8. Alexis has a collection of 75 different Beanie babies, 40 of which are bears. She decides to give five of them to her niece Gwen. If the selection is made randomly, what is the probability that Gwen gets at least four bears?

\[ \text{hypergeometric} \quad N = 75, \quad n = 40, \quad n = 5 \]

\[ P[X \geq 4] = P[X = 4] + P[X = 5] \]

\[ = \frac{\binom{40}{4} \binom{35}{5}}{\binom{75}{5}} + \frac{\binom{40}{5} \binom{35}{0}}{\binom{75}{5}} = .2235 \]
9. April is going to be a contestant on a game show. Within each round of the show, she will answer five questions, and in order to pass that round, at least four of them must be answered correctly. As soon as she fails any two rounds, the game will be over and she will go home. (The failed rounds do not have to be consecutive.)

The probability that April will answer any given question correctly is 89%, and you may assume that her chance of answering any one question correctly is independent of her chance of answering any other question correctly.

What is the probability that April will go home immediately following the seventh round?

(A) .000005
(B) .000007
(C) .000041
(D) .000053
(E) .005066
(F) .005608
(G) .033645
(H) .035465
(I) .1331
(J) .1428

First find the probability that she passes any given round.

\[
\text{binomial success: correct answer} \quad n = 5 \quad p = .89
\]

\[
P[X \geq 4] = 1 - P[X \leq 3] = 1 - \text{binomcdf}(5, .89, 3) = 1 - .096512 = .903488
\]

Then find the probability that she goes home immediately following the seventh round, in other words, that her second failed round is the seventh round.

\[
\text{negative binomial success: failed round} \quad r = 2 \quad \hat{p} = .096512
\]

\[
P[X = 7] = \binom{6}{1}(.903488)^5(.096512)^2 = .033645
\]

Part II. True-False (one point each)

Mark “A” on your answer card if the statement is true; mark “B” if it is false.

10. \( P(n, n) = n! \)  
    true  
    Both expressions represent the number of ways to select \( n \) items (from a pool of \( n \) items) and put them in order.

11. Suppose \( A \) and \( B \) are (nonempty) mutually exclusive events. Then \( P[A|B] = P[B|A] \).
    true  
    Both expressions are equal to zero.
12. Suppose that at Hitchcock High School, there are no students who are both on the football team and in the marching band. Then the events "The student is on the football team," and "The student is in the marching band," are independent.

\[
\text{false \quad These events are mutually exclusive, but they are not independent.}
\]

13. Suppose \( A \) and \( B \) are events with probabilities as shown in the following Venn diagram. Then \( A \) and \( B \) are independent.

\[
\begin{align*}
\text{true} & \quad P[A], P[B] = (0.8, 0.75) \\
& \quad = 0.6 = P[A \cap B]
\end{align*}
\]

14. The following is a valid density for a discrete random variable \( X \).

\[
\begin{array}{c|cccccc}
 x & 1 & 2 & 3 & 4 & \ldots \\
f(x) & \frac{1}{2} & \frac{1}{6} & \frac{1}{18} & \frac{1}{54} \\
\end{array}
\]

\[
\text{geometric series} \quad a = \frac{1}{2}, \quad r = \frac{1}{3}
\]

\[
\text{false} \quad \frac{a}{1-r} = \frac{\frac{1}{2}}{1 - \frac{1}{3}} = \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4} \neq 1
\]

15. Let \( X \) be a discrete random variable, and suppose \( E[X] = 7 \). Then \( E[X^2] = 49 \).

\[
\text{false} \quad E[X^2] \neq (E[X])^2
\]

16. The flip of a coin is an example of a Bernoulli trial.

\[
\text{true} \quad \text{There are only two outcomes.}
\]

17. Suppose \( X \) is a hypergeometric random variable with \( N = 200, r = 8, \) and \( n = 40 \). It is appropriate to approximate probabilities for \( X \) using the binomial distribution with \( p = 0.04 \).

\[
\text{false} \quad N \text{ fails to be at least 20 times as large as } n.
\]
Part III. Short Answer (one point each)

The answer to each of these is right or wrong: no work is required, and, with rare exceptions, no partial credit will be given. Give only one answer to each. (If you give more than one answer, the poorer one will count.)


$$P[A \cap B] = P[B|A] \cdot P[A] = (.6)(.8) = .48$$

19. Consider the discrete random variable $X$ whose density is given as follows. Find $E[X^2]$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>.4</td>
<td>.2</td>
<td>.1</td>
<td>.3</td>
</tr>
</tbody>
</table>

$$E[X^2] = (4)(.4) + (9)(.2) + (25)(.1) + (64)(.3) = 25.1$$

For problems #20 through #22, consider the discrete random variable $X$ such that $\mu_X = 20$ and $\sigma_X = 3$.

20. Find $\mu_{50-X}$.

$$\mu_{50-X} = 50 - \mu_X = 50 - 20 = 30$$

21. Find $\sigma_{50-X}$.

$$\sigma_{50-X} = \sqrt{1-1} \sigma_X = 3$$

22. Fill in the blank in the following statement:

According to Chebyshev's Inequality, $P[11 < X < 29] \geq \frac{8}{9}$

$$\mu - 3\sigma \rightarrow \mu + 3\sigma$$

23. Let $X$ be a geometric random variable with $p = .4$. Find $\text{Var} X$. Simplify your answer.

$$\text{Var} X = \frac{q}{p^2} = \frac{.6}{(.4)^2} = 3.75$$

24. Let $X$ be a binomial random variable with $n = 40$ and $p = .175$. Find the mean of $X$. Simplify your answer.

$$E[X] = np = (40)(.175) = 7$$
25. Let $X$ be a negative binomial random variable with $p = .8$ and $r = 6$. List the values of $x$ for which this random variable has nonzero probability.

$$X = 4, 7, 8, 9, \ldots$$

Part IV. Free Response (6 points)

Follow directions carefully, and show all the steps needed to arrive at your solution.

(6) 26. Suppose the following is a moment generating function for a random variable $X$.

$$m_X(t) = \frac{1+2t}{1-5t}$$

(a) Find $E[X]$.

$$\frac{d}{dt} m_X(t) = \frac{(1-5t)(2) - (1+2t)(-5)}{(1-5t)^2} = \frac{7}{(1-5t)^2}$$

$$E[X] = \left[\frac{d}{dt} m_X(t)\right]_{t=0} = 7$$

(b) Find $\text{Var } X$.

$$\frac{d^2}{dt^2} m_X(t) = -14(1-5t)^{-3}(-5) = 70(1-5t)^{-3}$$

$$E[X^2] = \left[\frac{d^2}{dt^2} m_X(t)\right]_{t=0} = 70$$

$$\text{Var } X = 70 - 49 = 21$$

(c) Find $\sigma_X$.

$$\sigma_X = \sqrt{21}$$