Examination Number 2

Answer all questions. Questions have equal weight.

1. Let \( \vec{v} \) be the vector field

\[
\vec{v} = (ax^2 - bx) \hat{i} + (y + z) \hat{j} + (z + y - bx) \hat{k}
\]

(a) For what values of \( a \) and \( b \) will all line integrals be independent of path?
(b) For these values of \( a \) and \( b \) find a potential function \( u \) for \( \vec{v} \) (i.e. \( \vec{v} = \nabla u \)).
(c) Evaluate

\[
\int_{A(0,0,0)}^{B(1,1,1)} \vec{v} \cdot d\vec{r}
\]

2. \( w(x, y, z) = 2x^2 + y^2 + xz \) is a scalar field

(a) What is the value of \( w \) at \( P(1, 2, 0) \)?
(b) Find the magnitude and direction of the greatest rate of change of \( w \) at \( P \).
(c) Find the equation of the plane tangent to \( w(x, y, z) = w(1, 2, 0) \) at \( P \).
(d) Find a direction \( \hat{u} \) such that the derivative of \( w \) at \( P \) in this direction is \( (1/2) \) the maximum rate of change of \( w \) at \( P \).

3. Let \( S \) be the parabolic surface \( z = y^2 \). \( S_1 \) is the part of \( S \) cut off by the elliptic cylinder

\[
(x/2)^2 + y^2 = 1
\]

Compute

\[
\int \int_{S_1} \vec{n} \cdot \vec{v} \ dA \quad \text{where} \quad \vec{v} = y \hat{j} + x \hat{k}
\]

4. Find all eigenvalues and eigenfunctions.

\[
\text{ODE: } t \frac{d^2 y}{dx^2} - \frac{dy}{dx} + 4\lambda t^3 y = 0.
\]

\( \text{BC's: } y(0) = 0, \ y(1) = 0. \)

Hint: Let \( x = t^2 \)

State the orthogonality property of the eigenfunctions.
\( F = (x^2 - yz) \mathbf{i} + (y + z) \mathbf{j} + (y + z - 6x) \mathbf{k} \)

\[
\nabla \times F = \begin{vmatrix} i & j & k \\ 2x & y + z & 2 \\ 2y - 6x & y + z & 0 \end{vmatrix} = \mathbf{a} (1 - 1) + \mathbf{b} (6 - 6) + \mathbf{c} (2) = 0
\]

\( \nabla \times F = 0 \) for all values of \( x, y, z \)

\( \frac{\partial F_1}{\partial y} = x - y \Rightarrow \nabla (x, y, z) = x^3 - 6x^2 + 8 \)

\( \frac{\partial F_2}{\partial x} = y + z \Rightarrow \nabla (x, y, z) = y + z + 2 \)

\( \frac{\partial F_3}{\partial y} < \frac{\partial F_2}{\partial x} \Rightarrow \nabla (x, y, z) = \frac{y + z + 2}{2} \)

\( \nabla (x, y, z) = x^3 - 6x^2 + 8 + y + z + \frac{y + z + 2}{2} \)

\( \nabla (x, y, z) = x^3 - 6x^2 + 8 + y + z + \frac{y + z + 2}{2} \) potential function

\[ I = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x^3 - 6x^2 + 8 + y + z + \frac{y + z + 2}{2} \end{bmatrix} = 0 - 4 + \frac{1}{2} + 1 + k - k = \frac{1}{2}
\]

\( w(1, 2, 0) = 3^2 + 4^2 + 4 \) \( P(1, 2, 0) \)

\( w(1, 2, 0) = 2 + 4 = 6 \)

\( \mathbf{w} = (4x + 2y) \mathbf{i} + (x - \frac{z}{2}) \mathbf{j} + (x + y + z) \mathbf{k} \)

\[ \mathbf{w} = (4x + 2y) \mathbf{i} + (x - \frac{z}{2}) \mathbf{j} + (x + y + z) \mathbf{k} \]

\[ |\mathbf{w}| = \sqrt{32} \] magnitude of greatest rate of flow

\[ \nabla \mathbf{w} = \frac{4\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{32}} \]

\( \mathbf{R} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \) point on tangent plane

\( \mathbf{P}_0 = 2 \mathbf{i} + \mathbf{j} \) point of tangency

\[ \mathbf{R} \cdot \nabla \mathbf{w} \bigg|_{\mathbf{R} = \mathbf{P}_0} = \mathbf{R} \cdot \nabla \mathbf{w} (\mathbf{P}) = 0 \]

\[ \nabla \mathbf{w} (\mathbf{P}) = \frac{4\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{32}} \]

\( -4x + 2y + z = 0 \) \( -4x + 2y + z = 0 \) equation of tangent plane

\( \mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 \) \( \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = 0 \) and \( \mathbf{u}_1 \cdot \mathbf{u}_2 = 1 \) arbitrary

\[ \mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 \] \( \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = 0 \) with arbitrary

\[ 2 \mathbf{F} = \nabla \mathbf{w} (\mathbf{P}) \cdot \mathbf{u} = \frac{1}{2} |\mathbf{w}| \]

\[ \frac{\partial y}{\partial x} = \frac{\mathbf{u}_1 + \mathbf{u}_2}{\mathbf{u}_3} \]

\[ \frac{\partial y}{\partial x} = \frac{\mathbf{u}_1 + \mathbf{u}_2}{\mathbf{u}_3} \] leads to a quadratic \( \frac{y^2}{2} + \frac{z^2}{3} \)
3. \[ S: \quad z = y^2 \]

S is the part of \( S \) cut off by the elliptic cylinder \( \left( \frac{x}{2} \right)^2 + z^2 = 1 \).

Compute \( I = \int \int_S \vec{v} \cdot \vec{n} \, dA \) where \( \vec{n} = y \hat{j} + x \hat{k} \)

on \( S \): \[ \vec{v} = x \hat{i} + y \hat{j} + z \hat{k} \]

\[ \vec{v}_x = \hat{i} \]

\[ \vec{v}_y = 1 + 2y \hat{k} \]

\[ \vec{n} = \vec{v}_x \times \vec{v}_y = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 2y \\ 0 & 0 & 1 \end{vmatrix} = (2y) \hat{k} \]

\[ \vec{n} \cdot \vec{v} \, dA = \vec{n} \cdot \left( x \hat{i} + y \hat{j} + z \hat{k} \right) \, dA = \int \int_S (2y) \, dA = 2 \int \int_S dA \]

\[ \vec{v} = x \hat{c} \cos \theta \quad x = 2r \cos \phi \]

\[ y = r \sin \theta \quad y = r \sin \phi \]

\[ I = \int \int_S \left( -2r^4 \cos^2 \theta + 2r \cos \theta \right) \, dA \]

\[ I = \frac{2\pi}{3} \int \int_S \left( -2r^4 \cos^2 \theta + 2r \cos \theta \right) \, dA \]

\[ I = \frac{2\pi}{3} \int_0^1 \int_0^1 \left( -2r^4 \cos^2 \theta + 2r \cos \theta \right) \, dr \, d\theta \]

\[ I = \frac{2\pi}{3} \int_0^1 \left[ -2r^5 \cos^2 \theta + 2r \sin \theta \right]_0^1 \, d\theta \]

\[ I = \frac{2\pi}{3} \int_0^1 \left( -2 \cos^2 \theta + 2 \sin \theta \right) \, d\theta \]

\[ I = \frac{2\pi}{3} \left[ -2 \sin 2\theta + 2 \cos \theta \right]_0^1 \]

\[ I = \frac{2\pi}{3} \left( -2 + 2 \right) \]

\[ I = \frac{2\pi}{3} \]

\[ \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial x \partial y} = 0 \]

\[ 8E \A: \quad \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial x \partial y} = 0 \]

\[ \frac{\partial y}{\partial x} = 0 \quad \frac{\partial y}{\partial y} = 0 \]

\[ y(x) = A x + B \]

\[ y(1) = 0 \]

\[ y(2) = 0 \]

\[ y(0) = A \]

\[ y(1) = B \]

\[ y(2) = 0 \]

\[ y''(x) = 0 \]

\[ y''(x) = 0 \]

\[ y'''(x) = 0 \]

\[ y''''(x) = 0 \]

\[ y''''(x) = 0 \]

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