Home Work One Solutions

1. (a) i. Norm of the vector \((1 \ 2 \ 1)\) is \(\sqrt{1+4+1}=\sqrt{6}\).
ii. Norm of the vector \((1 \ 1 \ -1)\) is \(\sqrt{1+1+1}=\sqrt{3}\).
iii. Norm of the vector \((2 \ 3 \ 0)\) is \(\sqrt{4+9+0}=\sqrt{13}\).

(b) The cosine of the angle \(\theta\) between any two vectors \(a\) and \(b\) is given as follows
\[
\cos \theta = \frac{a \cdot b}{\|a\| \|b\|}
\]

Angle between the vectors \((1 \ 2 \ 1)\) and \((1 \ 1 \ -1)\) is
\[
\cos^{-1} \frac{1+2-1}{\sqrt{6}\sqrt{3}} = \frac{2}{\sqrt{18}} = 61.87 \text{ degrees} = 1.0799 \text{ radians}
\]

Angle between the vectors \((1 \ 2 \ 1)\) and \((2 \ 3 \ 0)\) is
\[
\cos^{-1} \frac{2+6+0}{\sqrt{6}\sqrt{13}} = \frac{8}{\sqrt{78}} = 25.07 \text{ degrees} = 0.437 \text{ radians}
\]

Angle between the vectors \((1 \ 1 \ -1)\) and \((2 \ 3 \ 0)\) is
\[
\cos^{-1} \frac{2+3+0}{\sqrt{3}\sqrt{13}} = \frac{5}{\sqrt{39}} = 36.81 \text{ degrees} = 0.642 \text{ radians}
\]

(c) Let \(u\) be the vector \((3 \ 4 \ -5)\). We have \(\|u\| = \sqrt{3^2 + 4^2 + (-5)^2} = \sqrt{50}\). The required two vectors are \(\frac{u}{\|u\|}\) and \(-\frac{u}{\|u\|}\) given by
\[
\pm \left( \frac{3}{\sqrt{50}} \quad \frac{4}{\sqrt{50}} \quad \frac{-5}{\sqrt{50}} \right)
\]

(d) Let \(a\) and \(b\) be the two vectors \((1 \ 3 \ 9)\) and \((2 \ 4 \ -6)\) respectively.
\[
\|a\| = \sqrt{91}, \|b\| = \sqrt{56}, \|a + b\| = \sqrt{67}
\]

Moreover
\[
|a \cdot b| = 40 \text{ and } \|a - b\| = \sqrt{227}
\]

Clearly \(\sqrt{67} < \sqrt{91} + \sqrt{56}, 40 < \sqrt{91} \sqrt{56}\) and \(\sqrt{227} > \sqrt{91} - \sqrt{56}\)

2. (a) \[\|(1 \ 1)\| = \sqrt{1^2 + 1^2} = \sqrt{2}\]
Angle between \((1 \ 1)\) and \((1 \ 0)\) is
\[
\cos^{-1} \frac{1}{\sqrt{2}} = 45 \text{ degrees}
\]

(b) \[\|(2 \ -3)\| = \sqrt{2^2 + (-3)^2} = \sqrt{13}\]
Angle between \((2 \ -3)\) and \((1 \ 0)\) is
\[
\cos^{-1} \frac{2}{\sqrt{13}} = -56.3 \text{ degrees}
\]

(c) \[\|(\sqrt{3} \ 1)\| = \sqrt{3^2 + 1^2} = \sqrt{4} = 2\]
Angle between \((\sqrt{3} \ 1)\) and \((1 \ 0)\) is
\[
\cos^{-1} \frac{\sqrt{3}}{2} = 30 \text{ degrees}
\]
\[\|(5, 12)\| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13\]

Angle between \((5, 12)\) and \((1, 0)\) is
\[\cos^{-1} \frac{5}{13} = 67.38 \text{ degrees}\]

\[\|(-5, -12)\| = \sqrt{(-5)^2 + (-12)^2} = \sqrt{169} = 13\]

Angle between \((-5, -12)\) and \((1, 0)\) is
\[\cos^{-1} \frac{-5}{13} = -112.62 \text{ degrees}\]

3. (a) If the vector is \((a, b)\), it follows that \(a^2 + b^2 = 1\), \(\frac{b}{a} = \frac{1}{\sqrt{3}}\). Solving, we get \(a = \frac{\sqrt{3}}{2}\) and \(b = \frac{1}{2}\). The required unit vector is \(\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)\).

(b) The required vector is an unit vector making an angle of -30 degrees with respect to the \(x\)-axis. The vector is \((.866, -.5)\).

(c) The vector is \((\frac{3}{5}, -\frac{4}{5})\).

(d) The slope of the tangent vector is 4. A unit vector with slope 4 is given by \(\pm \left(\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right)\).

(e) The required unit normal vector is given by \(\left(-\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}}\right)\).

4. Assume that the cube is of length, breadth and height given by \(a\). It can be aligned in such a way that the diagonal is the vector \(d = (a, a, a)\), the face diagonal is the vector \(f = (a, a, 0)\) and the edge is the vector \(e = (a, 0, 0)\).

(a) Angle between \(d\) and \(e\) is given by \(\cos^{-1} \frac{\frac{a^2}{\sqrt{3}\times a}}{a^2} = 54.74 \text{ degrees}\).

(b) Angle between \(d\) and \(f\) is given by \(\cos^{-1} \frac{\frac{2a^2}{\sqrt{12}\times a\sqrt{2}}}{a^2} = 35.26 \text{ degrees}\).

(c) If \(a \neq 0\) and \(b \neq 0\), then the line \(ax + by + c = 0\) can be parameterized in vector as
\[\{u + tv : t \text{ is any real number}\}\]

and where \(u = (-\frac{a}{b}, 0)\) and \(v = (\frac{c}{a}, -\frac{c}{b})\). The vector \((a, b)\) is clearly perpendicular to the vector \(v\). If \(a = 0\), the line is given by \(by + c = 0\) which is parallel to the x-axis. The vector \((0, b)\) is clearly parallel to the x-axis. If \(b = 0\), the line is given by \(ax + c = 0\) which is parallel to the y-axis. The vector \((a, 0)\) is clearly parallel to the y-axis.

5. The vectors \(A\) and \(B\) are defined as follows:
\[A = (2, 5), \quad B = (3, -4)\]

We have \(\|A\| = \sqrt{29}\), \(\|B\| = 5\) and \(A.B = 6 - 20 = -14\). It follows that

(a) \(\text{proj}_A B = \frac{B.A}{\|A\|^2} A = -\frac{14}{29} (2, 5)\)

(b) \(\text{proj}_B A = \frac{A.B}{\|B\|^2} B = -\frac{14}{25} (3, -4)\)

(c) \(C = A - \text{proj}_B A = \frac{1}{25} (92, 69)\)

(d) \(C.B = \frac{1}{25} (92, 69). (3, -4) = \frac{1}{25} \times (276 - 276) = 0\).

Hence the vectors \(C\) and \(B\) are orthogonal to each other.