5.3 Constrained Geometric Programming

**Theorem 5.12** (Them 5.3.11) *(Extended Arithmetic-Geometric Mean Inequality)* Suppose \( x_i > 0, \delta_i \) are all positive or all zero, \( \lambda = \sum_{i=1}^{n} \delta_i. \)

\[
(\sum_{i=1}^{n} x_i)^{\lambda} \geq \lambda^\lambda \prod_{i=1}^{n} \left( \frac{x_i}{\delta_i} \right)^{\delta_i}
\]

where \( 0^0 = 1 \) and \( (x_i/0)^0 = 1. \)

Equality iff \( \delta_i \) are all zero, or \( \delta_i \) are all positive and \( x_i = \frac{\delta_i}{\lambda} (\sum_{i=1}^{n} x_i). \)

Proof: If \( \delta_i \) are all zero, it is easy to see the LHS=RHS.

Next assume \( \delta_i \) are all positive(note the theorem does not cover the case where some \( \delta_i \) positive and some zeros). Since

\[
\sum_{i=1}^{n} \frac{\delta_i}{\lambda} = 1,
\]

\[
\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \left( \frac{\delta_i}{\lambda} \right) \left( \frac{\lambda x_i}{\delta_i} \right)
\]

\[
\geq \prod_{i=1}^{n} \left( \frac{\lambda x_i}{\delta_i} \right)^{\frac{\delta_i}{\lambda}} (A-G)
\]

with equality holds iff

\[
\frac{\lambda x_1}{\delta_1} = \cdots = \frac{\lambda x_n}{\delta_n}
\]

\[
\implies (\sum_{i=1}^{n} x_i)^{\lambda} \geq \prod_{i=1}^{n} \left( \frac{\lambda x_i}{\delta_i} \right)^{\delta_i} = \lambda^\lambda \prod_{i=1}^{n} \left( \frac{x_i}{\delta_i} \right)^{\delta_i}
\]

If \( \lambda x_i/\delta_i = M, \)

\[
\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \frac{M \delta_i}{\lambda} = \frac{M}{\lambda} \sum_{i=1}^{n} \delta_i = M.
\]

\[
\implies x_i = \frac{M \delta_i}{\lambda} = \frac{\delta_i}{\lambda} \sum_{i=1}^{n} x_i \square
\]

**Definition 5.3** (Def 5.3.2)

\[
(GP) \quad \begin{cases} 
\min g_0(t) \\
\text{subject to} \\
g_1(t) \leq 1, \ldots, g_k(t) \leq 1 \\
t_1, \ldots, t_m > 0 
\end{cases}
\]

where \( g_i \) are posynomials.

**Example 5.8** (Eg 5.3.3)
• 400 cubic yards gravel.
• open box: \( t_1, t_2, t_3 \) (ends \( t_2 \times t_3 \), sides \( t_1 \times t_3 \), bottom \( t_1 \times t_2 \))
• sides/bottom: $10/square yard.
• ends: $20/square yard
• Box/trip: $.10
• Goal: min cost

Number of trips:

\[
\frac{400}{t_1 t_2 t_3}
\]

Cost of box for ferry:

\[
\frac{400}{t_1 t_2 t_3} (0.1) = \frac{40}{t_1 t_2 t_3}
\]

Cost of box:

\[
40t_2 t_3 \text{(for two ends)} + 20t_1 t_3 \text{(for two sides)} + 10t_1 t_2 \text{(for bottom)}
\]

Hence

\[
\min \frac{40}{t_1 t_2 t_3} + 40 t_2 t_3 + 20 t_1 t_3 + 10 t_1 t_2
\]

which can be solved by the techniques from ch2.

How about a variant of this problem? Assume the constraint:

• Sides and bottom made from scarp material of 4 square yards.

i.e.

\[
2t_1 t_3 + t_1 t_2 \leq 4
\]

and let us consider following constrained minimization problem:

\[
\implies \begin{cases} 
\min \ g_0(t) = \frac{40}{t_1 t_2 t_3} + 40 t_2 t_3 \\
\text{subject to} \quad g_1(t) = \frac{t_1 t_3}{2} + \frac{t_1 t_2}{4} \leq 1 
\end{cases}
\]

(Note \( g_0(t) \) does not contain the sides and bottom costs any more). For any \( \lambda > 0 \) and \( t \) s.t. \( g_1(t) \leq 1 \),

\[
[g_1(t)]^\lambda \leq 1
\]

Hence for any \( \delta_1 > 0, \delta_2 > 0 \) s.t. \( \delta_1 + \delta_2 = 1 \),

\[
g_0(t) \geq g_0(t)[g_1(t)]^\lambda = \\
= \left( \frac{40}{t_1 t_2 t_3} + 40 t_2 t_3 \right)[g_1(t)]^\lambda
\]
Let $\delta_3 > 0$, $\delta_4 > 0$ and $\lambda = \delta_3 + \delta_4$.

$$[g_1(t)]^\lambda = \left(\frac{t_1 t_3}{2} + \frac{t_1 t_2}{4}\right)^\lambda$$

$$\geq \lambda^\lambda((\frac{t_1 t_3}{2\delta_3})^{\delta_1}(\frac{t_1 t_2}{4\delta_4})^{\delta_2})(EAG)$$

$$= \lambda^\lambda((\frac{1}{2\delta_3})^{\delta_1}(\frac{1}{4\delta_4})^{\delta_2})t_1^{\delta_1}t_2^{\delta_2}t_3^{\delta_3}$$

$$\implies g_0(t) \geq \nu(\delta_1, \delta_2, \delta_3, \delta_4) = ((\frac{40}{\delta_1})^{\delta_1}(\frac{40}{\delta_2})^{\delta_2}(\frac{1}{2\delta_3})^{\delta_3}(\frac{1}{4\delta_4})^{\delta_4})(\delta_3 + \delta_4)^{\delta_3 + \delta_4}$$

provided that

$$\delta_1 + \delta_2 = 1 \quad (5.3)$$
$$\delta_3 + \delta_4 = \lambda \quad (5.4)$$
$$-\delta_1 + \delta_3 + \delta_4 = 0 \quad (5.5)$$
$$-\delta_1 + \delta_2 + \delta_4 = 0 \quad (5.6)$$
$$-\delta_1 + \delta_2 + \delta_3 = 0 \quad (5.7)$$
$$\delta_i > 0 \quad (5.8)$$

Solution: $\delta^* = (2/3, 1/3, 1/3, 1/3)$.

$$\implies \nu(\delta^*) = 60$$

$\nu(\delta^*)$ is the lower bound of $g_0(t)$ subject to $g_1(t) \leq 1$.

Q: How to make min value = lower bound?

A:

1. Equality in $A-G$:

$$x_1 = \frac{40}{\delta_1 t_1 t_2 t_3} = x_2 = \frac{40 t_2 t_3}{\delta_2}$$

2. Equality in $EA-G$:

$$x_3 = \frac{t_1 t_2}{2}, x_4 = \frac{t_1 t_2}{4}$$
$$x_3 = \frac{\delta_3}{\lambda}(x_3 + x_4), x_4 = \frac{\delta_4}{\lambda}(x_3 + x_4)$$
Let $K = \frac{x_3 + x_4}{\lambda}$.

\[ \Rightarrow \frac{x_3}{\delta_3} = \frac{x_4}{\delta_4} = K \]
\[ \Rightarrow \frac{t_1 t_3}{\frac{3}{3}} = \frac{t_1 t_2}{\frac{3}{3}} = K \]

3. Equality in $[g_1(t)]^\lambda \leq 1$. Since $\delta_i^*$ are all positive, $\lambda = \delta_3 + \delta_4 > 0$.

\[ \Rightarrow g_1(t) = 1 \]

Under the above conditions, $g_0(t) = 60$.

\[ \Rightarrow x_1 = \frac{40}{\frac{3}{3} t_1 t_2 t_3} = x_2 = \frac{40 t_2 t_3}{\frac{3}{3}} = 60 \]

\[ 1 = g_1(t) = \frac{t_1 t_3}{2} + \frac{t_1 t_2}{4} = \frac{1}{3} K + \frac{1}{3} K = \frac{2}{3} K \]
\[ \Rightarrow K = \frac{3}{2} \]

\[ \Rightarrow \begin{cases} t_1 t_2 t_3 = 1 \\ 2 t_2 t_3 = 1 \\ t_1 t_3 = 1 \\ t_1 t_2 = 2 \end{cases} \]

\[ \Rightarrow \begin{cases} \log t_1 + \log t_2 + \log t_3 = 0 \\ \log t_2 + \log t_3 = -\log 2 \\ \log t_1 + \log t_3 = 0 \\ \log t_1 + \log t_2 = \log 2 \end{cases} \]

\[ \Rightarrow \log t_1 = \log 2, \log t_2 = 0, \log t_3 = -\log 2 \]
\[ \Rightarrow t_1 = 2, t_2 = 1, t_3 = \frac{1}{2} \]

In general, let

\[ u_j(t) = c_j \prod_{i=1}^{m} t_i^{a_{ji}} \]

where $c_j > 0$.

\[ (GP) \begin{cases} \min \\ \text{subject to} \\ \begin{aligned} g_0(t) &= u_1(t) + \cdots + u_n(t) \\ g_i(t) &= u_{n_{i-1}+1}(t) + \cdots + u_{n_i}(t) \leq 1, i = 1, \cdots, k \\ t_1 > 0, \cdots, t_m > 0 \end{aligned} \end{cases} \]
5.3. CONSTRAINED GEOMETRIC PROGRAMMING

Let \( n_k = p \) (total numbers of terms of all \( g_i(i = 0, \ldots, k) \)).

\[
\begin{align*}
(DGP) \quad \max \nu(\delta) &= \prod_{j=1}^{p} (\frac{a_j}{\delta_j})^{\beta_j} \prod_{i=1}^{k} \lambda_i(\delta_i)^{\lambda_i(\delta)} \\
\text{subject to} \quad \sum_{i=1}^{n_0} \delta_i &= 1 \\
\alpha_{11} \delta_1 + \cdots + \alpha_{p1} \delta_p &= 0 \\
\vdots \\
\alpha_{1m} \delta_1 + \cdots + \alpha_{mp} \delta_p &= 0 \\
\delta_i > 0, \lambda_i &= \delta_{n+i-1+1} + \cdots + \delta_{n+i} - 1, i = 1, \ldots, p
\end{align*}
\]

Example 5.9 (Eg 5.3.4)

\[
\begin{align*}
\min \quad & t_1^2 t_2^3 t_3^2 \\
\text{subject to} \quad & \sum_{i=1}^{n_0} \delta_i = 1 \\
& \delta_1 + 3 \delta_2 - \frac{1}{2} \delta_3 = 0 \\
& -\delta_1 + \delta_2 - \frac{1}{2} \delta_3 = 0 \\
& 2 \delta_1 - \delta_2 + \delta_5 = 0 \\
& t_i > 0
\end{align*}
\]

dual variables: \( \delta_i(i = 1, \ldots, 5) \) (total numbers of terms).

\[
\nu(\delta) = (\frac{1}{\delta_1})^{\delta_1}(\frac{1}{2\delta_2})^{\delta_2}(\frac{1}{4\delta_3})^{\delta_3}(\frac{1}{4\delta_4})^{\delta_4}(\frac{1}{4\delta_5})^{\delta_5}(\delta_3 + \delta_4 + \delta_5)^{\delta_3 + \delta_4 + \delta_5}
\]

- one factor for each term.
- one factor \( \lambda_i^{\lambda_i} \)

for each constraint.

Solution: \( \delta = (1, -1/3 + r/6, r, -8/3 + r/3, -7/3 + r/3) \).

Theorem 5.13 (Them 5.3.5)

1. Suppose \( t \) is a feasible vector of (GP) and \( \delta \) is a feasible vector of (DGP).

\[
g_0(t) \geq \nu(\delta) \text{(Primal-Dual Inequality)}
\]

2. If (GP) is superconsistent and \( t^* \) is a solution for (GP), (DGP) has a solution \( \delta^* \) s.t.

\[
g_0(t^*) = \nu(\delta^*)
\]

\[
\delta^*_i = \begin{cases} \frac{u_i(t^*)}{g_0(t^*)} & i = 1, \ldots, n_0 \\ \lambda_i(\delta^*)u_i(t^*) & i = n_{j-1} + 1, \ldots, n_j, j = 1, \ldots, k \end{cases}
\]
Proof:

1. straightforward.

2. (GP) can be transformed into the convex programming:

\[
(GP)^* \left\{ \begin{array}{l}
\min \ h_0(x) \\
\text{subject to } \ h_i(x) - 1 \leq 0, \ i = 1, \ldots, k
\end{array} \right.
\]

where \( x_j = \ln t_j \) and for

\[
g(t) = \sum_{i=1}^{n} c_i t_1^{\alpha_{q1}} \cdots t_m^{\alpha_{qm}}, \ c_i > 0
\]

\[
h(x) = \sum_{i=1}^{n} c_i e^{\sum_{j=1}^{m} \alpha_{ij} x_j}
\]

Since (GP) is superconsistent, so is \((GP)^*\). KKT condition for \((GP)^*\),

- \( \lambda_i^* \geq 0, i = 1, \ldots, k. \)
- \( \lambda_i^*(h_i(x^*) - 1) = 0, i = 1, \ldots, k. \)
- \( \frac{\partial h_i}{\partial x_j}(x^*) + \sum_{i=1}^{k} \lambda_i^* \frac{\partial h_i}{\partial x_j}(x^*) = 0, j = 1, \ldots, m. \)

Note

\[
\frac{\partial h_i}{\partial x_j} = \frac{\partial h_i}{\partial t_j} \frac{dt_j}{dx_j} = \frac{\partial g_i}{\partial t_j} e^{x_j}
\]

\[
\Rightarrow \frac{\partial g_0}{\partial t_j}(x^*) + \sum_{i=1}^{k} \lambda_i^* \frac{\partial g_i}{\partial t_j}(x^*) = 0, j = 1, \ldots, m
\]

\[
\Rightarrow t_j^* \frac{\partial g_0}{\partial t_j}(x^*) + \sum_{i=1}^{k} \lambda_i^* t_j^* \frac{\partial g_i}{\partial t_j}(x^*) = 0, j = 1, \ldots, m
\]

Note

\[
u_q(t) = c_q t_1^{\alpha_{q1}} \cdots t_m^{\alpha_{qm}}
\]

\[
\Rightarrow t_j^* \frac{\partial u_q}{\partial t_j}(t^*) = c_q \alpha_{qj} t_1^{\alpha_{q1}} \cdots t_m^{\alpha_{qm}} = \alpha_{qj} u_q(t^*)
\]

\[
\Rightarrow t_j^* \frac{\partial g_i}{\partial t_j}(t^*) = \sum_{q=n_{r-1}+1}^{n_r} \alpha_{qj} u_q(t^*)
\]

\[
\Rightarrow 0 = \sum_{q=1}^{n_0} \alpha_{qj} u_q(t^*) + \sum_{r=1}^{k} \sum_{q=n_{r-1}+1}^{n_r} \lambda_i^* \alpha_{qj} u_q(t^*)
\]

\[
\Rightarrow 0 = \sum_{q=1}^{n_0} \alpha_{qj} \frac{u_q(t^*)}{g_0(t^*)} + \sum_{r=1}^{k} \sum_{q=n_{r-1}+1}^{n_r} \alpha_{qj} \left( \frac{\lambda^* u_q(t^*)}{g_0(t^*)} \right).
\]
Define
\[
\delta^*_q = \begin{cases} 
  \frac{u_q(t^*)}{g_0(t^*)} & q = 1, \ldots, n_0 \\
  \frac{\lambda_q u_q(t^*)}{g_0(t^*)} & q = n_r - 1 + 1, \ldots, n_r, r = 1, \ldots, k
\end{cases}
\]

\(\delta^*\) is a feasible vector of (DGP).

Note
\[
\lambda_r(\delta^*) = \sum_{q=n_r-1+1}^{n_r} \delta^*_q = \sum_{q=n_r-1+1}^{n_r} \lambda_q \frac{u_q(t^*)}{g_0(t^*)}
\]
\[
= \frac{\lambda_r}{g_0(t^*)} \sum_{q=n_r-1+1}^{n_r} u_q(t^*) = \lambda^*_r \frac{g_r(t^*)}{g_0(t^*)}
\]

Slackness condition:
\[
\lambda^*_r(g_r(t^*) - 1) = 0 \\
\implies \lambda^*_r g_r(t^*) = \lambda^*_r \\
\implies \lambda_r(\delta^*) = \frac{\lambda^*_r}{g_0(t^*)} \\
\implies \delta^*_q = \frac{\lambda^*_r u_q(t^*)}{g_0(t^*)} = \lambda_r(\delta^*) u_q(t^*)
\]

From Primal-Dual Inequality,
\[
g_0(t^*) \geq \nu(\delta^*)
\]

Q: Are they equal?

A: Yes. Recall Primal-Dual inequality is derived by A-G and EA-G. The equality if there are equalities for both A-G, EA-G and \([g_r(t)]^\lambda = 1\).

- A-G equality:
\[
\sum_{q=1}^{n_0} u_q(t) = \sum_{q=1}^{n_0} \delta_q \frac{u_q(t)}{\delta_q} = \prod_{q=1}^{n_0} \left( \frac{u_q(t)}{\delta_q} \right) \delta_q
\]

iff
\[
\frac{u_1(t)}{\delta_1} = \ldots = \frac{u_{n_0}(t)}{\delta_{n_0}}
\]

This is exactly equivalent to the definitions of \(\delta_q(q = 1, \ldots, n_0)\).

- \([g_r(t)]^\lambda = 1\) where \(\lambda_r = \sum_{q=n_r-1+1}^{n_r} \delta_q\). This is equivalent to the slackness condition:
\[
\lambda^*_r(g_r(t^*) - 1) = 0
\]

where \(\lambda^*_r\) is the Lagrange multiplier.

Proof: If \(\lambda^*_r = 0\)
\[
[g_r(t)]^0 = 1
\]

If \(\lambda^*_r > 0, g_r(t^*) = 1\).
\[
[1]^\lambda_r = 1 \square
\]
• EA-G equality: let \( \lambda_r = \lambda_r(\delta) \).

\[
\left[ \sum_{q=n_{r-1}+1}^{n_r} u_q(t)^{\lambda_r} \right] = \lambda_r^{n_r} \prod_{q=n_{r-1}+1}^{n_r} \frac{u_q(t)}{\delta_q} \delta_q
\]

iff \( \delta_q \) are all zero, or \( \delta_i \) are all positive and

\[
u(t) = \frac{\delta_q}{\lambda(\delta)}(\sum_{q=n_{r-1}+1}^{n_r} u_q(t)) = \frac{\delta_q}{\lambda} g_r(t)
\]

This is equivalent to the definitions of \( \delta^*_q(q = n_{r-1} + 1, \cdots, n_r) \).

Proof: \( \delta_q \) are all zero iff \( \lambda_r(\delta) = 0 \). This is equivalent to the definition. \( \delta_q \) are all positive and

\[
u(t) = \frac{\delta_q}{\lambda} \left( \sum_{q=n_{r-1}+1}^{n_r} u_q(t) \right) = \frac{\delta_q}{\lambda} g_r(t)
\]

From the slackness condition above, in this case \( \lambda(\delta) > 0 \), thus \( g_r(t) = 1 \).

\[
u(t) = \frac{\delta_q}{\lambda(\delta)}
\]

\( \iff \delta_q = \lambda(\delta) u_q(t)\)

This is the definitions of \( \delta^*_q \) for \( q = n_{r_1} + 1, \cdots, n_r \).

Remark: Them (5.13) is based on two results:

• A-G/EA-G inequalities: they lead to Primal-Dual inequality and \( g_0(t^*) = \nu(\delta^*) \).

• KKT condition: they lead to the definitions of \( \delta^* \).

If there is unique feasible vector \( \delta \) for (DGP) (e.g., (5.8)), KKT conditions will be sufficient. The unique feasible vector \( \delta \) will be \( \delta^* \) and then \( t^* \) can be determined from \( \delta^* \).

If there is more than one feasible vector \( \delta \), A-G/EA-G inequalities will be needed to determine \( \delta^* \) as \( \nu(\delta^*) = \min_{\delta \geq 0} \nu(\delta) \).

**Example 5.10** Example (5.9): Let us re-derive the (DGP) using KKT conditions.

\[
\begin{align*}
\min & \quad t_1 t_2^{-1} t_3^2 \\
\text{subject to} & \quad \frac{1}{2} t_1^2 t_2 t_3 - 1 \\
& \quad \frac{1}{4} t_1^{-1/2} + \frac{1}{4} t_2^{-1/2} + \frac{1}{4} t_3^2 \leq 1 \\
& \quad t_i > 0
\end{align*}
\]

\[(GP) \]
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Change variables: $t_i = e^{x_i}$.

\[
(P) \begin{cases}
\min g_0(x) = u_1(x) = e^{x_1-x_2+2x_3} \\
g_1(x) = u_2(x) = \frac{1}{2} e^{3x_1+x_2-x_3} \leq 1 \\
g_2(x) = u_3(x) + u_4(x) + u_5(x) = \frac{1}{4} e^{-x_1/2} + \frac{1}{4} e^{-x_2/2} + \frac{1}{4} e^{x_3} \leq 1
\end{cases}
\]

Assume $t^*$ is a solution to (GP). Then $x^* = (\ln t_1^*, \ln t_2^*, \ln t_3^*)$ will be a solution to (P). Since (P) is a superconsistent convex program (e.g. $x = (0, 0, 0)$ is a Slater point). Hence, KKT conditions assert that there exists $\lambda$ s.t.

1. $\lambda \geq 0$.

2. $\nabla g_0 + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 = 0$

3. $\lambda_i(g_i(x) - 1) = 0$, $i = 1, 2$.

Assume $u_q(x) = c_q e^{\sum_{i=1}^{m} \alpha_{qi} x_i}$.

\[
\Rightarrow \frac{\partial u_q}{\partial x_j} = \alpha_{qj} c_q e^{\sum_{i=1}^{m} \alpha_{qi} x_i} = \alpha_{qj} u_q
\]

KKT (2):

\[
\frac{\partial g_0}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} = 0, i = 1, 2, 3
\]

\[
\Rightarrow \begin{cases}
u_1 + 3\lambda_1 u_1 - \frac{1}{2}\lambda_2 u_3 = 0 \\
u_1 + \lambda_1 u_2 - \frac{1}{2}\lambda_2 u_4 = 0 \\
u_1 - \lambda_1 u_2 + \lambda_2 u_5 = 0
\end{cases}
\]

Divide by $g_0$:

\[
\Rightarrow \begin{cases}1 + 3\lambda_1 \frac{u_2}{g_0} - \frac{1}{2}\lambda_2 \frac{u_3}{g_0} = 0 \\
-1 + \lambda_1 \frac{u_2}{g_0} - \frac{1}{2}\lambda_2 \frac{u_4}{g_0} = 0 \\
2 - \lambda_1 \frac{u_2}{g_0} + \lambda_2 \frac{u_5}{g_0} = 0
\end{cases}
\]

Let

\[
\delta_1 = 1, \delta_2 = \lambda_1 \frac{u_2}{g_0}, \delta_3 = \lambda_2 \frac{u_3}{g_0}, \delta_4 = \lambda_2 \frac{u_4}{g_0}, \delta_5 = \lambda_2 \frac{u_5}{g_0}
\]

\[
(5.9)
\]

\[
\Rightarrow \begin{cases}
\delta_1 = 0 \\
\delta_1 + 3\delta_2 - \frac{1}{2}\delta_3 = 0 \\
-\delta_1 + \delta_2 - \frac{1}{2}\delta_4 = 0 \\
2\delta_1 - \delta_2 + \delta_5 = 0
\end{cases}
\]

In addition,

\[
\begin{cases}
\lambda_1(\delta) = \delta_2 = \lambda_1 \frac{u_1}{g_0} = \lambda_1 \frac{u_1}{g_0} \\
\lambda_2(\delta) = \sum_{i=3}^{5} \delta_i = \lambda_2 \frac{u_3 + u_4 + u_5}{g_0} = \lambda_2 \frac{u_2}{g_0}
\end{cases}
\]
Note $\delta$ defined by (5.9) also makes the EAG quality on $g_2$ since all $\delta_3 = \delta_4 = \delta_5 = 0$ if $\lambda_2 = 0$, or

$$\begin{align*}
\frac{u_3}{\delta_3} &= \frac{u_4}{\delta_4} = \frac{u_5}{\delta_5} = \frac{g_0}{\lambda_2}
\end{align*}$$

In other words, $\delta^*$ defined by (5.9) satisfies

$$g_0(t^*) = \nu(\delta^*)$$

KKT condition (3) implies

$$\lambda_i g_i = \lambda_i, i = 1, 2.$$  

$$\implies \lambda_i(\delta) = \frac{\lambda_i}{g_0}$$

In other words, $\lambda_i(\delta)$ gives a quantitative measure of the sensitivity of the solution of the constrained geometric program to its $i$th constraint.

Three questions:

1. Q1: How do we know the existence of $t^*$?

2. Q2: How do we know the existence of $\delta^*$?

3. Q3: How do we know that $g(t^*) = \nu(\delta^*)$?

Q1 and Q2 can be answered by following Theorem: if (GP) and (DGP) are both consistent, there exist solutions $t^*$ to (GP) and $\delta^*$ to (DGP).

For our example here, both (GP) and (DGP) (e.g. $t = (1, 1, 1)$ is feasible for (GP) and $\delta = (1, [-1/3 + r/6, r, -8/3 + r/3, -7/3 + r/6], r \geq 14)$ is feasible for (DGP)) are consistent. Hence there exist solutions $t^*$ to (GP) and solutions $\delta^*$ to (DGP).

Q3 can be answered by Theorem (5.13): since (GP) is superconsistent (e.g. $t = (1, 1, 1)$ is a Slater point).