Experiment 6: Transmission Line Pulse Response

Lossless Distributed Networks

When the time required for a pulse signal to traverse a circuit is on the order of the rise or fall time of the pulse, it is no longer possible to approximate the circuit with lumped elements of ordinary circuit theory. Instead, it is necessary to treat the circuit components as having spatially distributed resistance, capacitance, and inductance. Practically, this is required in situations where the excitation frequency is high, the circuit element has a long physical dimension, and/or short pulse rise times are used. In this experiment, we will examine the properties of pulse propagation in long interconnects, or transmission lines.

Wave Propagation

![Diagram of a lossless transmission line](image)

Figure 1 – Infinitesimal section of a lossless transmission line

Our analysis of a distributed network begins by considering the infinitesimal section of Figure 1. In this analysis we assume for simplicity that the network has no series resistance or shunt conductance. With this assumption, we maintain the essential features of pulse propagation on transmission lines, while avoiding the complications of the conversion of rectangular pulses into gaussian or complementary error function pulse shapes due to the line losses. The circuit parameters in distributed network analysis are taken on a per unit dimension basis. That is, L in Figure 1 has units of Henries/meter and C has units of Farads/meter. On this basis, Kirchhoff’s current law gives

\[ I(x, t) = I(x + dx, t) + Cdx \frac{\partial V(x + dx, t)}{\partial t} \quad (1) \]

Or, using the definition of a derivative,

\[ \frac{\partial I(x, t)}{\partial x} = -C \frac{\partial V(x, t)}{\partial t} \quad (2) \]

Kirchhoff’s voltage law gives

\[ V(x, t) = V(x + dx, t) + Ldx \frac{\partial I(x, t)}{\partial t} \quad (3) \]

or

\[ \frac{\partial V(x, t)}{\partial x} = -L \frac{\partial I(x, t)}{\partial t} \quad (4) \]
Taking the partial derivative of equation (2) with respect to $t$ and the partial derivative of equation (4) with respect to $x$ (and the converse), we obtain

$$\frac{\partial^2 V(x,t)}{\partial x^2} = LC \frac{\partial^2 V(x,t)}{\partial t^2}$$  \hspace{1cm} (5)

$$\frac{\partial^2 I(x,t)}{\partial x^2} = LC \frac{\partial^2 I(x,t)}{\partial t^2}$$  \hspace{1cm} (6)

Equations (5) and (6) are referred to as wave equations.

**Propagation Velocity**

It is straightforward to verify by direct substitution in equations (5) and (6) that general solutions of the equations are

$$V(x,t) = V^+(t - \frac{x}{v}) + V^-(t + \frac{x}{v})$$  \hspace{1cm} (7)

$$I(x,t) = I^+(t - \frac{x}{v}) + I^-(t + \frac{x}{v})$$  \hspace{1cm} (8)

where

$$v = \frac{1}{\sqrt{LC}}$$  \hspace{1cm} (9)

The functions with the $+$ superscript describe waves traveling in the $+x$ direction while those with the $-$ superscript represent waves traveling in the $-x$ direction. The velocity of propagation is $v$.

If we examine the parameters of a coaxial cable, for example, the capacitance/meter is

$$C = \frac{2\pi \epsilon}{\ln(b/a)}$$  \hspace{1cm} (10)

where $\epsilon$ is the permittivity of the dielectric, $a$ is the radius of the inner wire, and $b$ is the distance from the center of the inner wire to the inner edge of the outer sheath. The inductance/meter is essentially the external inductance which is

$$L = \frac{\mu}{2\pi} \ln(b/a)$$  \hspace{1cm} (11)

where $\mu$ is the magnetic permeability of the material between the inner and outer conductors. In the product $LC$, geometric factors cancel and we have

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu \epsilon}} = \frac{c}{\sqrt{\mu_r \epsilon_r}}$$  \hspace{1cm} (12)

where $\mu_r$ is the relative magnetic permeability of the insulating medium between the conductors, $\epsilon_r$ is its relative dielectric permittivity, $c$ is the velocity of electromagnetic propagation in free space, $3 \times (10)^8$ meters/sec., and $v$ is the velocity of the propagating wave in the cable. These relations hold true for all two conductor uniform cross-section transmission lines.
**Characteristic Impedance**

We consider the relation between a current wave, \( I^+(t - x/v) \), and a voltage wave \( V^+(t - x/v) \), both traveling in the +x direction. For these waves

\[
\frac{\partial}{\partial t} = -v \frac{\partial}{\partial x}
\]

(13)

So, from equation (2) we have

\[
\frac{\partial I^+(x,t)}{\partial x} = C_v \frac{\partial V^+(x,t)}{\partial x}
\]

(14)

which implies that waves of current and voltage traveling in the +x direction have the same functional dependence on the variable \( t - x/v \), and differ only by a scale factor, \( C_v \).

We can write

\[
I^+(t - \frac{x}{v}) = C_v V^+(t - \frac{x}{v})
\]

(15)

And since the units of \( C_v \) are Ohms\(^{-1} \), then further,

\[
I^+(t - \frac{x}{v}) = \frac{1}{R_0} V^+(t - \frac{x}{v})
\]

(16)

In the same way,

\[
I^-(t + \frac{x}{v}) = -\frac{1}{R_0} V^-(t + \frac{x}{v})
\]

(17)

The quantity, \( R_0 \), is the ‘characteristic impedance’ of the transmission line.

\[
R_0 = \sqrt{\frac{L}{C}} = \frac{1}{vC} = vL
\]

(18)

So, the general solutions of the wave equations for the transmission line are

\[
V(x,t) = V^+(t - \frac{x}{v}) + V^-(t + \frac{x}{v})
\]

(19)

\[
I(x,t) = \frac{1}{R_0} V^+(t - \frac{x}{v}) - \frac{1}{R_0} V^-(t + \frac{x}{v})
\]

(20)

Notice from the above equations that the characteristic impedance is not an impedance in the usual sense that it is measurable by an ohmmeter, but is simply a characteristic parameter for a distributed network. That is, \( R_0 \) is equal to \( V(x,t) \) divided by \( I(x,t) \) only in certain special conditions.
The characteristic impedance of a coaxial cable is found by inserting equations (10) and (11) into equation (18) with the result

\[
R_0 = \frac{1}{2\pi} \sqrt{\frac{\mu_r}{\varepsilon_r}} \ln\left(\frac{b}{a}\right) = 60\sqrt{\frac{\mu_r}{\varepsilon_r}} \ln\left(\frac{b}{a}\right)
\]

We will be using a RG 58/U coaxial cable in the experiment. This cable has \(\mu_r = 1\), \(\varepsilon_r = 2.3\), and \(R_0 = 50\) Ohms.

**Reflection Coefficients**

**Infinite Line**

![Figure 2 – Semi-infinite transmission line](image)

The particular solutions to the wave equations (19) and (20) are determined by the boundary conditions on the distributed network. Consider the semi-infinitely long transmission line shown in Figure 2. Because it extends to infinity, there can be no waves traveling in the \(-x\) direction. From equations (19) and (20)

\[
V(x,t) = V^+(t - \frac{x}{v})
\]

(22)

\[
I(x,t)R_0 = V^+(t - \frac{x}{v})
\]

(23)

and

\[
V(x,t) = I(x,t)R_0
\]

(24)

for all values of \(x \geq 0\) and \(t\). This is one of the special cases where the characteristic impedance, \(R_0\), is actually a resistance. From Kirchhoff’s voltage law and equation (24)

\[
V(0,t) = \frac{R_0}{R_s + R_0} V_s(t)
\]

(25)

which is just a voltage divider equation. So, at the sending end the semi-infinite transmission line appears as a resistor whose value is the characteristic impedance.
Terminated Line

In a finite transmission line, we have to keep all terms of equations (19) and (20) to allow for possible reflections. An examination of Figure 3 shows that

\[ R_L = \frac{V(L, t)}{I(L, t)} = R_0 \frac{V^+(t - \frac{L}{v}) + V^-(t + \frac{L}{v})}{V^+(t - \frac{L}{v}) - V^-(t + \frac{L}{v})} \]  

(26)

Defining a load reflection coefficient as the ratio of the reflected voltage wave to the incident voltage wave gives

\[ \rho_L = \frac{V^-(t + \frac{L}{v})}{V^+(t - \frac{L}{v})} \]  

(27)

And equation (26) becomes

\[ R_L = R_0 \frac{1 + \rho_L}{1 - \rho_L} \]  

(28)

which gives

\[ \rho_L = \frac{R_L - R_0}{R_L + R_0} \]  

(29)

for an incident voltage wave on the receiving or load end.

Using a similar analysis, the source reflection coefficient is

\[ \rho_s = \frac{R_s - R_0}{R_s + R_0} \]  

(30)
for an incident wave on the sending or source end. In these analyses, we have not examined reflection coefficients for current waves but they are a straightforward extension of the voltage wave arguments.

Notice in equations (29) and (30) that when $R_L$ and $R_S$ are equal to the characteristic impedance, $R_0$, the reflection coefficients on both ends of the line are zero. Therefore, there are no reflected waves on the transmission line and the line is said to be properly terminated. This is also true when one end has a zero reflection coefficient and the line is energized from the other end through a source with a non-zero reflection coefficient. Any other situation will give a reflected wave on the line. For example, a zero impedance termination or short circuit will have a reflection coefficient of $-1$ and the reflected wave will exactly cancel the incident wave at the short circuit termination. An open circuit termination has a reflection coefficient of $+1$ and so the reflected wave adds to the incident wave at the open circuit termination. In this latter case, the reflected current wave cancels the incident current wave at the termination so as to give zero current through the open circuit. The resulting waveforms at the sending and receiving ends of a transmission line can be most easily determined with the aid of reflection diagrams.

**Reflection Diagrams and Voltage Wave Forms**

![Reflection Diagram](image)

Figure 4 – $R_L$ and $R_S = R_0$

Figure 4 shows the reflection diagram, input, and output voltage waveforms for a line terminated at both generator and load ends with $R_0$. Because of voltage division, $\frac{1}{2}$ of the input signal, $+V_S$, travels down the line and reaches the load after a time equal to the time delay of the line, which is $T$.

$$T = \frac{L}{v}$$

At the load, the reflection coefficient is zero and there is no reflected wave.
In Figure 5, $+V_S/2$ propagates down the line and reaches the short circuit in time $T$. With a reflection coefficient of $-1$, $-V_S/2$ is immediately reflected, canceling the incident wave of $+V_S/2$, leaving 0 Volts at the load. The reflected wave of $-V_S/2$ travels back to the source and cancels the initial $+V_S/2$ at time $2T$.

In Figure 6, $+V_S/2$ propagates down the line and reaches the open circuit at time $T$. With a reflection coefficient of $+1$, $+V_S/2$ is immediately reflected, adding to the incident $+V_S/2$ and leaving $V_S$ Volts at the load. The reflected wave of $+V_S/2$ travels back to the source and adds to the initial $+V_S/2$ at time $2T$.

Figure 7 – Capacitor as load with $R_S = R_0$
The reflection diagram and wave forms associated with an uncharged load capacitance are indicated in Figure 7. An uncharged capacitor has no voltage across it and so appears to a pulse as an initial short circuit. A fully charged capacitor has the maximum circuit voltage across it and appears to a pulse as an open circuit. Considering these effects, the reflection coefficient for an uncharged load capacitor goes from $-1$ to $+1$ with a time constant of $R_0C$.

Justification for the time constant $R_0C$ comes from applying Thevenin’s theorem to the transmission line, generator, and generator resistance ($R_0$) as seen from the capacitor. Capacitive terminations are common in high speed digital integrated circuits.

![Figure 8 - Inductor as load with $R_S = R_0$](image)

The reflection diagram and wave forms associated with an uncharged load inductor are indicated in Figure 8. An uncharged inductor has no initial current through it and so appears to a pulse initially as an open circuit. A fully charged inductor has the maximum current through it and so appears to a pulse as a short circuit. The reflection coefficient for an uncharged load inductor therefore goes from $+1$ to $-1$ with a time constant of $L/R_0$.

![Figure 9 - Diode as load with $R_S = R_0$](image)

The reflection diagram and waveforms associated with a diode or base-emitter junction of a bipolar transistor load can be seen in Figure 9. The resistance and the reflection coefficient for these nonlinear devices varies with the voltage change that occurs during the finite rise time of the pulse. Junction capacity also plays a part. For these devices, the reflection coefficient will have the form

$$\rho_L = \frac{kT}{qI} - \frac{R_0}{kT + R_0}$$

(32)
where

\[ I = I_0 \exp\left(\frac{qV}{kT}\right) \]  

(33)
**Experiment**

**Equipment List**

1. Printed Circuit Board with SN74LS241 Octal Buffer/Driver
1. Printed Circuit Board Fixture
1. Coil of RG58/U Coaxial Cable with $50 \, \Omega$ Characteristic Impedance, $\varepsilon_r = 2.30$.
1. Short circuit termination
1. $50\,\Omega$ Termination
1. $93 \, \Omega$ Termination
1. 2 nF Termination
1. 4.7 $\mu$H Termination
1. Blue Termination
1. Blue I Termination

**Procedure**

![Expt. No. 9
Printed Circuit Board](image)

Figure 9 – Octal Buffer/Driver Circuit for Coaxial Cable Transmission Lines

Set the HP function generator to a 20 kHz square wave with an output that varies from 0 to a positive value of approximately 4V and turn it off. Set a DC power supply to a positive value of 5V and turn it off. Now make the connections to the octal buffer/driver indicated in Figure 9 with the HP function generator applied to the input and turn the dc power supply and the function generator on. Observe and copy the 5, 50, and 10 k$\Omega$ output signals from the buffer/driver. A short (1 meter) 50 $\Omega$ coaxial cable may be used. For proper operation these signals should be square waves with amplitude of approximately 0.4 V and period of 50 $\mu$s.

(a) Connect a long (coiled) 50 $\Omega$ coaxial cable to the 50 $\Omega$ output of the buffer/driver with provisions for measuring the input and output waveforms of the coaxial cable on the oscilloscope. Make some marking on the coil you use in this project so that you can identify it and use the same coil for project 7. A short (1 meter) 50 $\Omega$ coaxial cable may be used to measure inputs to the long cable. Trigger the oscilloscope on the rising edge of the input waveform at a time scale of around 50 ns per division and experiment with 10X, 1X probes, a short (1 meter) coaxial cable, and a tee connection to see which gives the best output waveforms.
Observe and copy the magnitudes and time relationships between the input and output waveforms for long (coiled) cable terminations of open circuit, short circuit, 50 Ω, 93 Ω, 2 nF, 4.7 μH, blue, and blue I.

(b) Connect the long (coiled) 50 Ω coaxial cable to the 5 Ω output of the buffer/driver. Observe and copy the magnitudes and time relationships between the input and output waveforms for long (coiled) cable terminations of open circuit, short circuit, 50 Ω, and 93 Ω.

(c) Connect the long (coiled) 50 Ω coaxial cable to the 10 kΩ output of the buffer/driver. Observe and copy the magnitudes and time relationships between the input and output waveforms for an open circuit long (coiled) cable termination.

Report

(a) (i) Explain how you are able to determine the characteristic impedance of the long (coiled) coaxial cable transmission line from your measurements. Calculate the impedance of this transmission line using your measurements for each of the source impedance’s in parts (a), (b), and (c)

(ii) Explain how you are able to determine the length of the long (coiled) coaxial cable transmission line from your measurements. What did you obtain for its value?

(iii) Calculate the capacitance and inductance per unit length of the long (coiled) coaxial cable transmission line.

(iv) Present and compare the theoretically expected and experimental magnitudes and time relationships between the input and output waveforms for output terminations of open circuit, short circuit, 50 Ω, 93 Ω, 2 nF, and 4.7 μH with a 50 Ω source resistance.

(v) From your experimental data derive equivalent circuits with component values for blue and blue I terminations.

(b) Present and compare the theoretically expected and experimental magnitudes and time relationships between the input and output waveforms for long (coiled) coaxial cable terminations of open circuit, short circuit, 50 Ω, and 93 Ω with a 5 Ω driver output resistance.

(c) Present and compare the theoretically expected and experimental magnitudes and time relationships between the input and output waveforms for an open circuit long (coiled) coaxial cable termination with a 10 kΩ driver output resistance. Show your calculation in detail.

(d) Consider the junction between properly terminated 50 Ω and 93 Ω transmission lines. Design a resistive network that will eliminate reflections from the junction back into either line. (Note: There will be some loss of power at the matching network.)

References and Suggested Reading