1 Computing Information Rate Functions

In these problems, you will derive the basic forms of the information rate functions for several distributions of interest.

1.1 Gaussian Densities with Different Means

Define the two hypotheses:

\[ H_1 \]: \( R_i = m_i + W_i, \quad i = 1, 2, \ldots, n \)

\[ H_0 \]: \( R_i = W_i, \quad i = 1, 2, \ldots, n \)

where the random variables \( W_i \) are i.i.d. \( \mathcal{N}(0, \sigma^2) \). The means \( m_i \) under hypothesis \( H_1 \) are known.

In class, we derived the log-likelihood ratio test for this problem. When written in vector notation, the loglikelihood ratio equals

\[ l(r) = \frac{1}{\sigma^2} (r - \frac{1}{2} m)^T m, \quad (1) \]

where \( m \) is the \( n \times 1 \) vector of the means \( m_i \).

a. Follow our derivation in class to verify that the performance is completely determined by a signal-to-noise ratio

\[ d^2 = \sum_{i=1}^{n} \frac{m_i^2}{2\sigma^2} \quad (2) \]

b. Compute the moment generating function \( \Phi_0(s) \) for the log-likelihood ratio given hypothesis \( H_0 \).

c. Compute the information rate function in two ways. First, directly from the log-moment generating function \( \phi_0(s) = \ln \Phi_0(s) \), compute

\[ I_0(\gamma) = \max_s s\gamma - \phi_0(s). \quad (3) \]

Second, use the relative entropy formula and the tilted density function. Recall that the tilted density function is defined by

\[ p(r : s) = \frac{p_{R}(r|H_0)e^{s l(r)}}{\Phi_0(s)}. \quad (4) \]
The second formula for $I_{0}(\gamma)$ is then

$$I_{0}(\gamma) = D(p(r : s)||p_{R}(r|H_{0})).$$ \hfill (5)

Show that the two expressions are equal. This requires substituting into the relative entropy formula the value of $s$ such that the desired mean of $l(R)$ is achieved (the desired mean is the threshold $\gamma$).

### 1.2 Gaussian Densities with Different Variances

Consider the hypothesis testing problem with hypotheses:

$$H_{1}: \ R_{i} \text{ is } N(0, \sigma_{1i}^{2}), \ i = 1, 2, \ldots, n$$

$$H_{0}: \ R_{i} \text{ is } N(0, \sigma_{0i}^{2}), \ i = 1, 2, \ldots, n,$$

and under either hypothesis, the random variables $R_{i}$ are independent.

**a.** Derive the log-likelihood ratio test for this problem. Show that the test statistic is

$$l(r) = \frac{1}{2} \sum_{k=1}^{n} \ln \frac{\sigma_{1k}^{2}}{\sigma_{0k}^{2}} - r_{k}^{2} \left( \frac{1}{\sigma_{1k}^{2}} - \frac{1}{\sigma_{0k}^{2}} \right).$$ \hfill (6)

**b.** Find the tilted probability density function for this problem. Show that this tilted density function corresponds to independent Gaussian random variables with zero mean and variances $\sigma_{sk}^{2}$, where

$$\frac{1}{\sigma_{sk}^{2}} = \frac{1 - s}{\sigma_{0k}^{2}} + \frac{s}{\sigma_{1k}^{2}}.$$ \hfill (7)

**c.** Find the log-moment generating function $\phi_{0}(s)$. Recall (as in the last problem) that $\Phi(s)$ is the normalizing factor for the tilted density function and $\phi_{0}(s) = \ln \Phi_{0}(s)$.

**d.** For this problem, it may be difficult to find $s$ explicitly as a function of the threshold $\gamma$. In order to circumvent this difficulty, the standard approach is to represent the curves parametrically as a function of $s$. Find $\gamma$ as a function of $s$ by using the property that the mean of the log-likelihood function using the tilted density equals $\gamma$.

**e.** Using the representation of $\gamma$ from part d, the information rate function may be found as a function of $s$ using

$$I_{0}(\gamma(s)) = s\gamma(s) - \phi_{0}(s),$$ \hfill (8)

and the result from part c.

### 1.3 Poisson Distribution Functions

Consider the hypothesis testing problem with hypotheses:

$$H_{1}: \ R_{i} \text{ is Poisson with means } \lambda_{1i}, \ i = 1, 2, \ldots, n$$

$$H_{0}: \ R_{i} \text{ is Poisson with means } \lambda_{0i}, \ i = 1, 2, \ldots, n,$$

and under either hypothesis, the random variables $R_{i}$ are independent.

**a.** Derive the log-likelihood ratio test for this problem. Show that the test statistic is

$$l(r) = \sum_{k=1}^{n} r_{k} \ln \frac{\lambda_{1k}}{\lambda_{0k}} - \lambda_{1k} + \lambda_{0k}.$$ \hfill (9)
b. Find the tilted probability density function for this problem. Show that this tilted density function corresponds to independent Poisson random variables with means $\lambda_{sk}$, where

$$\ln[\lambda_{sk}] = (1 - s) \ln[\lambda_{0k}] + s \ln[\lambda_{1k}]$$

(10)

c-e. Repeat parts c, d, and e from the last problem here. This is needed again since there is no straightforward expression for $s$ in terms of $\gamma$.

2 Basic Estimation Theory

Suppose that $X_1, X_2, \ldots, X_n$ is a set of zero mean Gaussian random variables. The covariance between any two random variables is

$$E\{X_i X_j\} = 0.9^{\mid i - j \mid}.$$  

(11)

a. Derive an expression for $E\{X_2^2 \mid X_1\}$.

b. Derive an expression for $E\{X_3 \mid X_2, X_1\}$.

c. Find the conditional mean squared error $E\{(X_3 - E\{X_3 \mid X_2, X_1\})^2\}$.

(12)

d. Can you generalize this to a conclusion about the form of $E\{X_n \mid X_1, X_2, \ldots, X_{n-1}\}$?

3 Problem from Exam 1, 1990

Suppose a $2N \times 1$ real valued Gaussian random vector $s$ is observed, where $s$ is $\mathcal{N}(0, K_s)$, and where $K_s$ is a block diagonal matrix with $N$ blocks of size $2 \times 2$ each

$$K_s = \begin{bmatrix}
\sigma_s^2 & \rho \sigma_s^2 & 0 & 0 & \cdots & 0 & 0 \\
\rho \sigma_s^2 & \sigma_s^2 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \sigma_s^2 & \rho \sigma_s^2 & \cdots & 0 & 0 \\
0 & 0 & \rho \sigma_s^2 & \sigma_s^2 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & \sigma_s^2 & \rho \sigma_s^2 \\
0 & 0 & 0 & 0 & \cdots & \rho \sigma_s^2 & \sigma_s^2
\end{bmatrix}.$$  

(13)

The two parameters in this covariance matrix are $\sigma_s^2$ and $\rho$.

a. Find the maximum likelihood estimates for $\sigma_s^2$ and $\rho$.

b. Determine a lower bound for the variance in estimating $\sigma_s^2$.

**Hint:** You may want to use the results of the problem from Problem Set 2, where you derived the loglikelihood ratio for the binary detection problem

$$H_0: \ r = n$$  

(14)

$$H_1: \ r = s + n$$  

(15)

where $n$ is a $2N \times 1$ real-valued Gaussian random vector which is 0 mean with covariance matrix $\sigma_n^2 I$ ($n$ is $\mathcal{N}(0, \sigma_n^2 I)$) and the signal vector $s$ is independent of $n$ and is a $2N \times 1$ real-valued Gaussian random vector, $\mathcal{N}(0, K_s)$. 