Midterm Exam ESE 405

October 21, 2009

Your Name: Solutions

Your ID #: ________________

There are 4 problems in this examination. You must show details of your work to receive full credit.
1. This problem is about the binomial and Poisson probability distribution functions.

**Binomial Probability: “Wheel of Fortune”**

A player bets on one of the numbers 1 through 6. Three dice are then rolled, and if the number bet by the player appears \( i \) times \((i = 1, 2, 3)\), then the player wins \( i \) units; on the other hand, if the number bet by the player does not appear on any of the dice, then the player loses 1 unit. Determine the expected gain by the player in this game.

The number of times that the number bet appears is a binomial random variable with parameters \((n, p) = (3, \frac{1}{6})\).

Let \( X \) be the player’s gain in the game.

\[
\begin{align*}
P(X = -1) &= \binom{3}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^3 = \frac{125}{216} \\
P(X = 1) &= \binom{3}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^2 = \frac{75}{216} \\
P(X = 2) &= \binom{3}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^1 = \frac{15}{216} \\
P(X = 3) &= \binom{3}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^0 = \frac{1}{216}
\end{align*}
\]

\[
E(X) = (-1) \left(\frac{125}{216}\right) + (1) \left(\frac{75}{216}\right) + (2) \left(\frac{15}{216}\right) + (3) \left(\frac{1}{216}\right) = \frac{-17}{216}
\]

**Poisson Probability: Occurrence of Earthquakes**

Suppose that earthquakes occur in the western portion of the United States in accordance with the Poisson probability with rate \( \lambda = 2 \) per week. Find the probability that at least 3 earthquakes occur during the next 2 weeks. Find the probability distribution of the time, starting from now, until the next earthquake.

Let \( X \) be the number of earthquakes during the next 2 weeks.

\[
P(X = k) = \frac{e^{-\lambda t} (\lambda t)^k}{k!} = \frac{e^{-2 \cdot 2} (2 \cdot 2)^k}{k!} = \frac{e^{-4} (4)^k}{k!}
\]

\[
P(X \geq 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2)
= 1 - e^{-4} - 4e^{-4} - \frac{4^2}{2} e^{-4} - \frac{4^3}{6} e^{-4} = 1 - 13e^{-4} \approx 0.76
\]

Let \( Y \) be the amount of time in weeks until the next earthquake.

\[
F(t) = P(Y \leq t) = 1 - e^{-\lambda t} = 1 - e^{-2t}
\]

Probability density of \( Y = f(t) = \frac{d}{dt} F(t) = 2e^{-2t} \)
2. Samples of certain products are drawn randomly and \( n (=20) \) measurements of their weights are made as shown below. Estimate the mean, standard deviation, and variance from the sample data. Two sheets of probability plot scales (one for the uniform probability and another for the normal probability). Plot the data in the two probability plots, and discuss which of the two probabilities better describes the weight of the product.

<table>
<thead>
<tr>
<th>Sample Number</th>
<th>Weight (Pounds)</th>
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<tbody>
<tr>
<td>1</td>
<td>9.63</td>
</tr>
<tr>
<td>2</td>
<td>9.86</td>
</tr>
<tr>
<td>3</td>
<td>10.20</td>
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<tr>
<td>4</td>
<td>10.48</td>
</tr>
<tr>
<td>5</td>
<td>9.82</td>
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<td>6</td>
<td>10.07</td>
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<td>7</td>
<td>10.39</td>
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<td>8</td>
<td>10.03</td>
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<td>9</td>
<td>9.34</td>
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<td>10</td>
<td>10.26</td>
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<tr>
<td>17</td>
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<tr>
<td>18</td>
<td>10.13</td>
</tr>
<tr>
<td>19</td>
<td>9.96</td>
</tr>
<tr>
<td>20</td>
<td>9.75</td>
</tr>
</tbody>
</table>

Sample mean \( \bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} = 10.0065 \)

Sample Variance \( S_X^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1} = 0.1117 \)

Sample standard deviation \( S_X = \sqrt{S_X^2} = 0.3342 \)

See the graphs for the plots. Using all 20 data points, neither graph looks good. By eliminating the "outlier" points for each graph, the uniform plot seems a better fit.
3. The height of men is known to be normally distributed, but the mean and standard deviation are unknown. The heights of 20 randomly selected men were measured and the sample mean is determined to be 69.80 inches and the sample standard deviation is 3.1 inches. Find a 95% confidence interval for the mean height of all men.

Since the standard deviation (and the variance) is unknown, we use the t-distribution.

The confidence interval using the t-distribution is given as

$$
\bar{X} - t_{\alpha/2, n-1} \frac{S}{\sqrt{n}} \leq \mu \leq \bar{X} + t_{\alpha/2, n-1} \frac{S}{\sqrt{n}}
$$

\(n=20\), \(\alpha=0.05\), \(\bar{X}=69.80\), \(S=3.1\)

From the t-distribution table (Appendix IV),
we get \(t_{0.025, 19} = 2.093\)

\[
\Rightarrow 69.80 - 2.093 \left( \frac{3.1}{\sqrt{20}} \right) \leq \mu \leq 69.80 + 2.093 \left( \frac{3.1}{\sqrt{20}} \right)
\]

\[
68.35 \leq \mu \leq 71.25
\]
4. A manufacturer sells a product with the specification value of 16 mm in diameter. The diameter of the product is normally distributed and the standard deviation is known to be 3 mm. A single sample was selected by a customer from a large lot and its diameter was measured to be 11.5 mm. Can the customer claim that the product’s true average diameter is less than the specification value from this sample? (Let $\alpha = 0.05$.)

**Hypothesis Testing**

$H_0: \mu = 16$

$H_1: \mu < 16$

$x = 11.5 \implies z = \frac{x - \mu}{\sigma} = \frac{11.5 - 16}{3} = -1.5$

From table in Appendix:

$z_{1.05} = 1.645$

Thus, $H_0$ is rejected if $z < -1.645$. Since this is not true, the customer cannot claim that the product’s true average diameter is less than the specification value.