1.4 Scalar and Vector Fields

A field is a description of how a physical quantity varies in the region of the field (and with time).

(a) Scalar fields

Ex: Depth of a lake, \( d(x, y) \)

Temperature in a room, \( T(x, y, z) \)

Can be depicted graphically by image intensities or by constant magnitude contours or surfaces

(b) Vector Fields

Ex: Force field in three dimensions

\[
F(x, y, z) = F_x(x, y, z)\mathbf{a}_x + F_y(x, y, z)\mathbf{a}_y + F_z(x, y, z)\mathbf{a}_z
\]

Depicted graphically by constant magnitude contours or surfaces, and direction lines (or stream lines).

(c) Static Fields Fields not varying with time.

(d) Dynamic Fields Fields varying with time, e.g., temperature in a room, \( T(x, y, z; t) \)

Example: Linear velocity field of points on a rotating disk

\[
v(x, y) = v_x(x, y)\mathbf{a}_x + v_y(x, y)\mathbf{a}_y
\]
D1.10 For \( T(x, y, z, t) \), find shapes of the constant temperature surfaces \((T_0, \text{a constant})\) at \( t=0 \)

\[- T_0 \left[ x(1 + \sin \pi t) \right]^2 + \left[ 2y(1 - \cos \pi t) \right]^2 + 4z^2 \]

(a) \( T(x, y, z, 0) = T_0 \left[ x(1 + 0) \right]^2 + \left[ 2y(1 - 1) \right]^2 + 4z^2 \)

\[ T_0 (x^2 + 4z^2) \]

Constant temperature surfaces are elliptic cylinders, \( \{x^2 + 4z^2\} = \text{const} \).

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Finding the equation for the direction lines of a vector field

The direction lines of a vector field are curves formed tangential to the field.

\[
\mathbf{a} \times \mathbf{F} = \begin{vmatrix}
\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\
dx & dy & dz \\
F_x & F_y & F_z 
\end{vmatrix} = 0
\]

\[
dx = \frac{dy}{F_y} = \frac{dz}{F_z}
\]

Similarly

\[
\frac{dr}{F_r} = \frac{r \, d\phi}{F_\phi} = \frac{dz}{F_z} \quad \text{cylindrical}
\]

\[
\frac{dr}{F_r} = \frac{r \, d\theta}{F_\theta} = \frac{r \sin \theta \, d\phi}{F_\phi} \quad \text{spherical}
\]
\textbf{P1.26 (b) } x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z  \\
\text{(Position vector)}
\[
\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}
\]
\[
\ln x = \ln y + \ln C_1 = \ln z + \ln C_2
\]
\[
\ln x = \ln C_1 = \ln C_2 z
\]
\[
x = C_1 y = C_2 z
\]

\therefore Direction lines are straight lines emanating radially from the origin. For the line passing through (1, 2, 3),
\[
1 = C_1 (2) = C_2 (3)
\]
\[
\therefore C_1 = \frac{1}{2}, \quad C_2 = \frac{1}{3}
\]
\[
x = \frac{y}{2} = \frac{z}{3}
\]
or, \(6x = 3y = 2z\)

\textbf{Example: } Linear velocity field of points on a rotating disk \(\mathbf{v}(x, y) = v_r (x, y) \mathbf{a}_r + v_\phi (x, y) \mathbf{a}_\phi\)

\[
\mathbf{v}(r, \phi) = \omega r \mathbf{a}_\phi
\]
\[
\frac{dr}{0} = \frac{r \, d\phi}{\omega r}
\]
\[
\frac{dr}{0} = \frac{r}{\text{constant}}
\]
In Cartesian coordinates,
\[ \mathbf{v}(x, y) = v_x(x, y)\mathbf{a}_x + v_y(x, y)\mathbf{a}_y \]
\[ \mathbf{v}(x, y) = \omega \mathbf{r} \cdot \mathbf{a}_r + \omega \mathbf{r} \cdot \mathbf{a}_\phi \]
\[ = \omega \mathbf{r} \cdot (-\sin \phi \mathbf{a}_x + \cos \phi \mathbf{a}_y) \]
\[ = \omega (-y \mathbf{a}_x + x \mathbf{a}_y) \]
\[ \frac{dx}{dy} \frac{dy}{dx} \quad -y \quad x \]
\[ x \, dx + y \, dy = 0 \]
\[ x^2 + y^2 = \text{constant} \]

**Review Questions**

1.21. Discuss briefly your concept of a scalar field and illustrate with additional examples.
1.22. Discuss briefly your concept of a vector field and illustrate with additional examples.
1.23. How do you depict pictorially the gravitational field of the earth?
1.24. Discuss the procedure for obtaining the equations for the direction lines of a vector field.

**1.5 Sinusoidally Time-Varying Fields**

Polarization: an interesting and relevant example of time-varying fields

*(Elements of Eng EM, Sec. 3.6)*
The polarization of a sinusoidally time-varying field describes how the position of the tip of the field vector at a given point in space varies with time.

**Linear Polarization:**
Tip of the vector describes a line.

**Circular Polarization:**
Tip of the vector describes a circle.

**Elliptical Polarization:**
Tip of the vector describes an ellipse.

(i) **Linear Polarization**
\[ \mathbf{F}_1 = F_1 \cos(\omega t + \phi) \mathbf{a}_x \]
Magnitude varies sinusoidally with time.
\[ \therefore \text{Linearly polarized in the } x \text{ direction.} \]
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\[ F_2 = F_2 \cos (\omega t + \theta) \mathbf{a}_y \]

Magnitude varies sinusoidally with time
Direction remains along the y axis

; ; Linearly polarized in the y direction.

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If two (or more) component linearly polarized vectors are in phase, (or in phase opposition), then their sum vector is also linearly polarized.

Ex: \[ F = F_1 \cos (\omega t + \phi) \mathbf{a}_x + F_2 \cos (\omega t + \phi) \mathbf{a}_y \]

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(ii) **Circular Polarization**

If two component linearly polarized vectors are
(a) equal in amplitude
(b) differ in direction by 90°
(c) differ in phase by 90°,
then their sum vector is circularly polarized.
Example:

\[ F = F_1 \cos \omega t \mathbf{a}_x + F_2 \sin \omega t \mathbf{a}_y \]

\[ |F| = \sqrt{(F_1 \cos \omega t)^2 + (F_2 \sin \omega t)^2} \]

\[ = F_0, \text{ constant} \]

\[ \alpha = \tan^{-1} \frac{F_2 \sin \omega t}{F_1 \cos \omega t} \]

\[ = \tan^{-1} \left( \tan \omega t \right) = \omega t \]

\[ \mathbf{F}_1 = F_0 \cos \left(2\pi \times 10^3 t - 2\pi z\right) \mathbf{a}_x \]

\[ \mathbf{F}_2 = F_0 \cos \left(2\pi \times 10^3 t - 3\pi z\right) \mathbf{a}_y \]

\( \mathbf{F}_1 \) and \( \mathbf{F}_2 \) are equal in amplitude (= \( F_0 \)) and differ in direction by 90°. The phase difference (say \( \phi \)) depends on \( z \) in the manner \(-2\pi z - (-3\pi z) = \pi z\).

(a) At \((3, 4, 0), \phi = \pi (0) = 0\).

\( (\mathbf{F}_1 + \mathbf{F}_2) \) is linearly polarized.

(b) At \((3, -2, 0.5), \phi = \pi (0.5) = 0.5 \pi\).

\( (\mathbf{F}_1 + \mathbf{F}_2) \) is circularly polarized.

Review Questions (revisit in Ch. 3)

1.25. A sinusoidally time-varying vector is expressed in terms of its components along the \( x \)-, \( y \)-, and \( z \)-axes. What is the polarization of each of the components?

1.26. What are the conditions for the sum of two linearly polarized sinusoidally time-varying vectors to be circularly polarized?

1.27. What is the polarization for the general case of the sum of two sinusoidally time-varying linearly polarized vectors having arbitrary amplitudes, phase angles, and directions?

1.28. Considering the seconds hand on your analog watch to be a vector, state its polarization. What is the frequency?
1.6 The Electric Field (section 1.5 in text)

The electric field is a force field acting on charges by virtue of the property of charge.

Coulomb’s Law

\[ F_1 = \frac{Q_1 Q_2}{4\pi \varepsilon_0 R^2} \hat{a}_{21} \]
\[ F_2 = \frac{Q_1 Q_2}{4\pi \varepsilon_0 R^2} \hat{a}_{12} \]

\( \varepsilon_0 \) = permittivity of free space

\[ \varepsilon_0 \approx 10^{-9} \text{ F/m} \]

\[ \approx \frac{1}{36\pi} \text{ F/m} \]

D1.13(b) For point charges at vertices of square, find electric force on each charge

From the construction, it is evident that the resultant force is directed away from the center of the square. The magnitude of this resultant force is given by

\[ 2 \frac{Q^2}{4\pi \varepsilon_0 \left(2a^2\right)} \cos 45^\circ + \frac{Q^2}{4\pi \varepsilon_0 \left(4a^2\right)} \]

\[ = \frac{1}{\sqrt{2}a^2} + \frac{1}{4a^2} \]

\[ = 0.957 \frac{N}{a^2} \]
Electric Field Intensity, $E$

is defined as the force per unit charge experienced by a small test charge when placed in the region of the field.

$$E = \lim_{q \to 0} \frac{F}{q}$$

Thus

$$F_q = qE$$

Units: $\frac{N}{C} = \frac{N \cdot m}{C \cdot m} = \frac{V}{m}$

Sources: Charges; Time-varying magnetic field

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Electric Field Intensity, $E$

See text for applications:

- Electrostatic separation of minerals (separation based on particle charge)
- Cathode ray tube operation (variable horizontal and vertical deflection of electron beam)

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Electric Field of a Point Charge

$$F = \frac{Qq}{4\pi \varepsilon_0 R^2} \hat{a}_s$$  (Coulomb’s Law)

$$= q \left( \frac{Q}{4\pi \varepsilon_0 R^2} \hat{a}_s \right)$$

$$= q \left( E \text{ due to } Q \right)$$

$$\therefore E \text{ due to } Q = \frac{Q}{4\pi \varepsilon_0 R^2} \hat{a}_s$$
Constant magnitude surfaces are spheres centered at \( Q \). Direction lines are radial lines emanating from \( Q \).

**E due to charge distributions**

(a) **Collection of point charges**

\[
E = \sum_{j=1}^{n} \frac{Q_j}{4\pi\varepsilon_0 R_j^2} \hat{a}_j
\]

(b) **Line Charges**

Line charge density, \( \rho_L \) (C/m)

(c) **Surface Charges**

Surface charge density, \( \rho_S \) (C/m²)

(d) **Volume Charges**

Volume charge density, \( \rho \) (C/m³)

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**E1.3 Finitely-Long Line Charge**

\[
\rho_L = \rho_L = 4\pi\varepsilon_0 \ C/m
\]

\[
dE = 2\frac{\rho_L}{4\pi\varepsilon_0} \frac{dz}{r^2 + z^2} \cos \alpha \hat{a}_z
\]
\[
\begin{align*}
\text{For } a \to \infty, \ E & \to \frac{\rho_0}{2\pi \varepsilon_0} \mathbf{a}_r,
\end{align*}
\]
Infinite plane sheets at \( z = 0, 2 \) and 4 with uniform surface charge densities. Given

\[
\begin{align*}
E(3,5,1) &= 0 \, \text{V/m} \\
E(1,-2,3) &= 6a_\sigma \, \text{V/m} \\
E(3,4,5) &= 4a_\sigma \, \text{V/m}
\end{align*}
\]

find charge densities and \( E(-2,1,-6) \).

\[
\begin{align*}
\frac{1}{2\varepsilon_0} (\rho_{31} - \rho_{32} - \rho_{33}) &= 0 \\
\frac{1}{2\varepsilon_0} (\rho_{31} + \rho_{32} - \rho_{33}) &= 6 \\
\frac{1}{2\varepsilon_0} (\rho_{31} + \rho_{32} + \rho_{33}) &= 4
\end{align*}
\]

Solving, we obtain

(a) \( \rho_{31} = 4\varepsilon_0 \, \text{C/m}^2 \)  (b) \( \rho_{32} = 6\varepsilon_0 \, \text{C/m}^2 \)

(c) \( \rho_{33} = -2\varepsilon_0 \, \text{C/m}^2 \)  (d) \( E(-2,1,-6) = -4a_\sigma \, \text{V/m} \)

**Review Questions**

1.29. State Coulomb’s law. To what law in mechanics is Coulomb’s law analogous?
1.30. What is the value of the permittivity of free space? What are its units?
1.31. What is the definition of electric field intensity? What are its units?
1.32. Describe the electric field due to a point charge. How do you determine the electric field due to a charge distribution?
1.33. Discuss the different types of charge distributions. How do you determine the electric field due to a charge distribution?
1.34. Describe the electric field due to an infinitely long line charge of uniform density.
1.35. Describe the electric field due to an infinite plane sheet of uniform surface charge density.