ESE 251 Presentation
by
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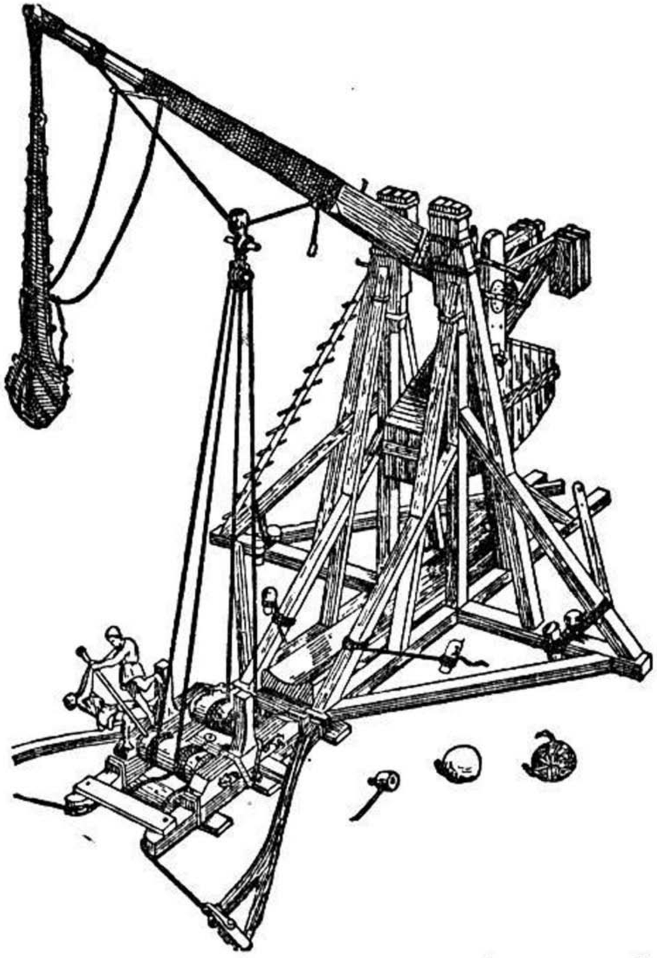
A Mathematical Model for a Trebuchet

History

- A trebuchet is a medieval siege engine that uses a massive counterweight to accurately propel a projectile great distances

- The trebuchet was thought to have been invented in China, and introduced to Europe during the 13th century



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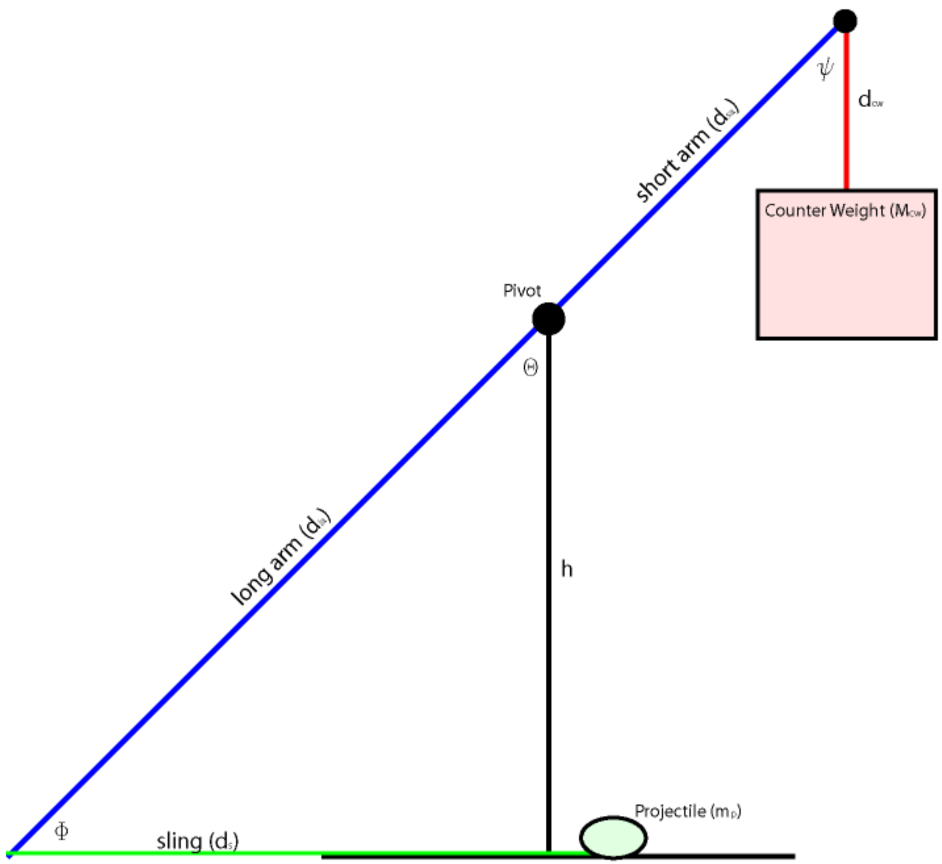
Trebuchet Geometry



- Constants:
  - M_{cw}: mass of the counterweight
  - m_p: mass of the projectile
  - h: height of the pivot
  - ds: sling length
  - dcw: CW length
  - dsa: length of the short arm
  - dla: length of the long arm
- State Variables:
  - θ: pivot angle
  - ϕ: sling angle
  - ψ: pivot angle

Background Analysis

- A trebuchet is a device that converts potential energy to kinetic energy

$$Mgh_{cw} \rightarrow \frac{1}{2} mv^2{}_{projectile}$$

- From basic physics we know that the range of a projectile with initial velocity v and angle α is

$$R = \frac{2v^2 \sin\alpha \cos\alpha}{g}$$

- Thus, the maximum theoretical range of a trebuchet is given by

$$R_{\max} = 2\frac{M_{cw}}{m_p} h$$

Problem Formulation

- Given a trebuchet of fixed dimensions, you wish to design an electronic release mechanism

- For a desired release angle α, you want to find the time at which the control mechanism should release
  - i.e. find the time t such that sling angle $\phi=\alpha$.

Model Assumptions

- Assume that all structures are rigid, and that the device is fixed to the ground
- Assume that all surfaces are smooth, and all contacts well lubricated, so frictional effects are negligible
- Assume that the arm beam has negligible mass
- For simplicity, suppose that the given trebuchet has the counterweight fixed to the arm, so $d_{cw}=0$

Modeling Strategy

- Split the model into two cases
  - i) The projectile slides along a smooth trough
  - ii) The projectile swings unconstrained through the air
- Use Lagrange's equations to derive equations of motion
- Use a numerical solver to solve the equations of motion

Lagrangian Mechanics

- Let T denote kinetic energy and V denote the potential energy of a system. The Lagrangian of the system is defined as

$$L = T - V$$

- For each coordinate qi, Lagrange's equation is

$$0 = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial q_i}$$

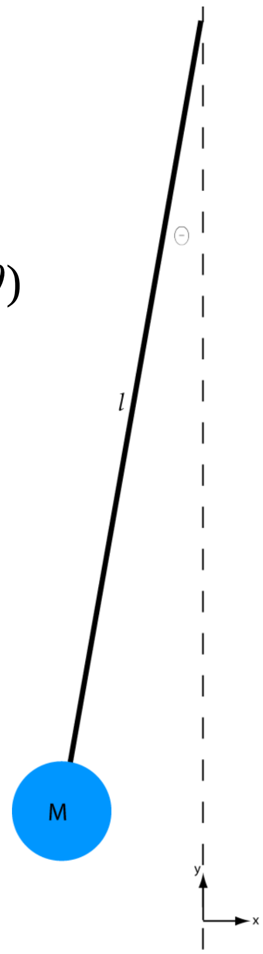
- Ex: Consider a simple pendulum of length *l* and mass *m*

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$V = mgl(1 - \cos\theta)$$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos\theta)$$

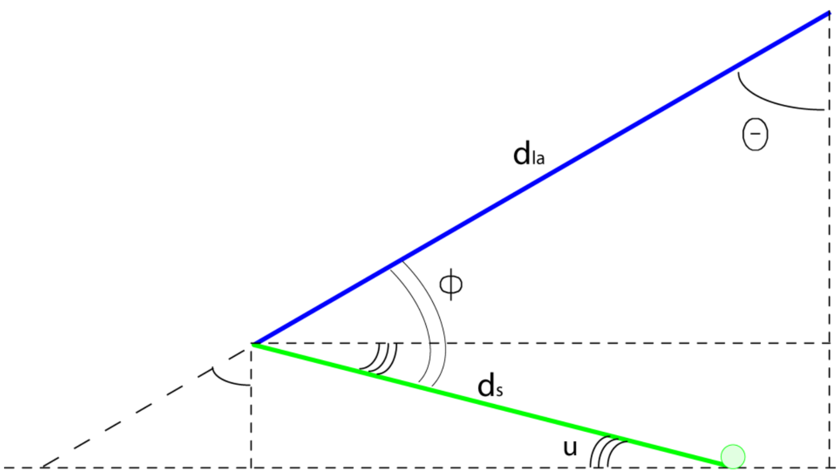
$$0 = \ddot{\theta} - \frac{g}{l}\sin\theta$$



Case (i): Constrained Sling

- During case (i), the sling is constrained to move along a given curve
- With a little geometry, we can derive the following constraint

$$f(\theta,\phi)=\phi+\theta-\frac{\pi}{2}-\sin^{-1}\left[\frac{h-d_{la}\cos\theta}{d_s}\right]$$



- To model this case, adapt Lagrange's equation by a Lagrange multiplier

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right]-\frac{\partial L}{\partial q_i}-\lambda a_{q_i}=0 \qquad a_{q_i}=\frac{\partial f}{\partial q_i}$$

- It can then be shown that

$$a_\theta=1-\frac{d_{la}\sin(\theta)}{\sqrt{d_s^{\,2}-\left(h-d_{la}\cos(\theta)\right)^2}} \qquad a_\phi=1$$

- Case (i) ends when the net force in the y direction is zero. This yields the following.

$$g=-\ddot{x}_p\left[\frac{h-d_{la}\cos\theta}{d_s^{\,2}-(h-d_{la}\cos\theta)^2}\right]$$

Case (ii): Free Sling

- We are interested in the dynamics of the system in terms of θ and ψ
  - We need to solve for the coordinates in terms of the two angles

$$x_c = d_{sa}\sin\theta \qquad x_p = d_s\sin(\theta+\phi) - d_{la}\sin(\theta)$$

$$\dot{x}_c = d_{sa}\cos\theta\cdot\dot{\theta} \qquad \dot{x}_p = [d_s\cos(\theta+\phi) - d_{la}\cos(\theta)]\dot{\theta} + d_s\cos(\theta+\phi)\cdot\dot{\phi}$$

$$y_c = h + d_{sa}\cos\theta \qquad y_p = h + d_s\cos(\theta+\phi) - d_{la}\cos(\theta)$$

$$\dot{y}_c = -d_{sa}\sin\theta\cdot\dot{\theta} \qquad \dot{y}_p = [-d_s\sin(\theta+\phi) + d_{la}\sin(\theta)]\dot{\theta} - d_s\sin(\theta+\phi)\cdot\dot{\phi}$$

- We then plug these expressions into the Cartesian expressions for kinetic and potential energy

$$T = \frac{1}{2}\left[M_{cw}(\dot{x}_c{}^2 + \dot{y}_c{}^2) + m_p(\dot{x}_p{}^2 + \dot{y}_p{}^2)\right] \qquad V = M_{cw}g\cdot y_c + m_p g\cdot y_p$$

Initial Conditions

- We assume that the trebuchet has the standard initial configuration given to the right

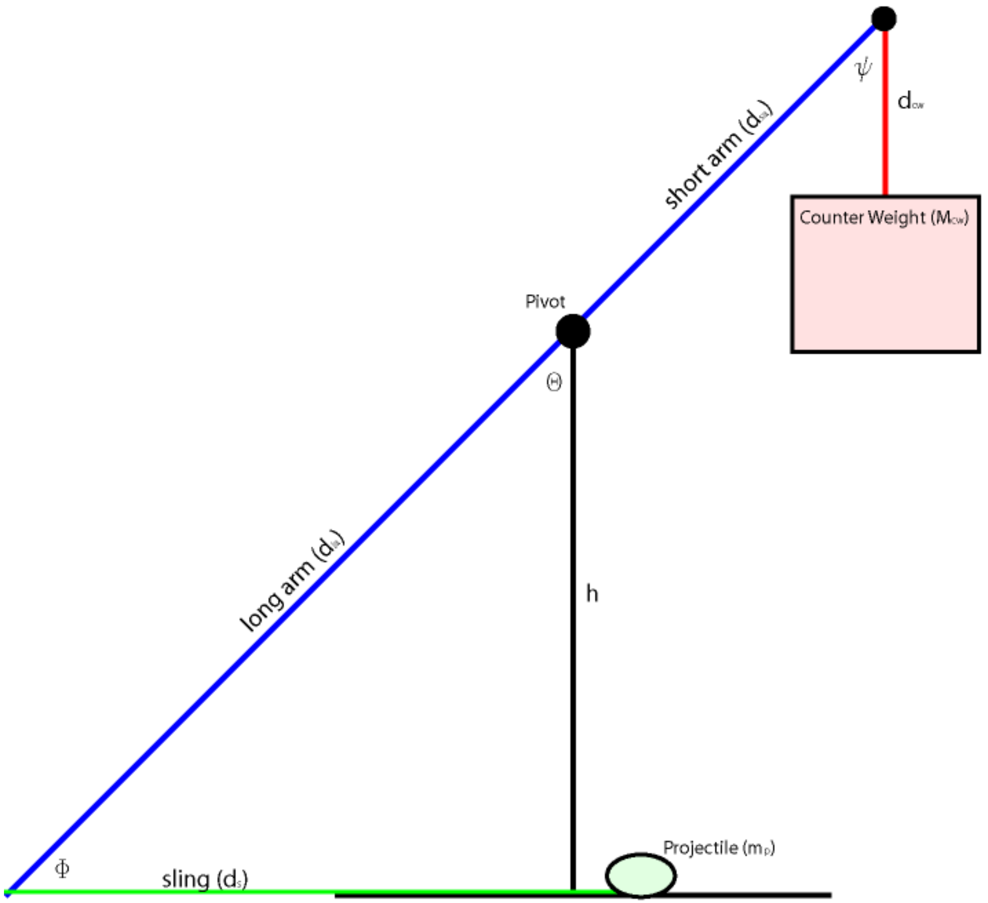
$$\sin\phi_i = \frac{h}{d_{la}}$$

$$\theta_i = \frac{\pi}{2} - \phi_i$$

- All initial velocities are identically zero

- Note: to be physically possible we must have

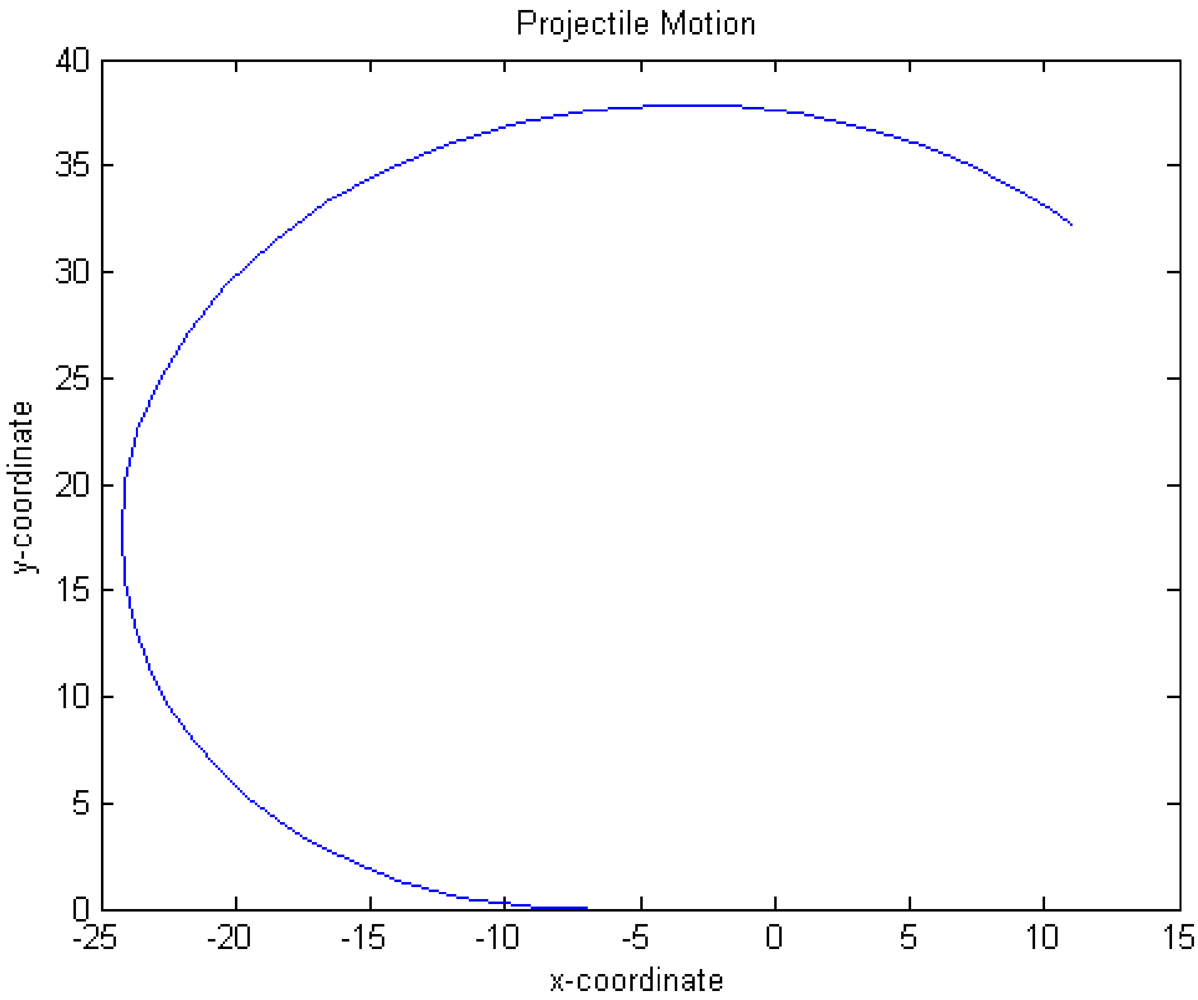
$$d_{sa} + d_{cw} < h$$



Results

- To solve the ODE we use the MATLAB ode45 function
  - Uses a 4^{th} order Runge-Kutta numerical integrator
  - Must put equations in the form y'=f(t,y) to solve
- Release time:
  - For given dimensions, a desired release time was found to be 1.217 seconds after triggering
  - Expected range of the device was ~85ft

Visualizations



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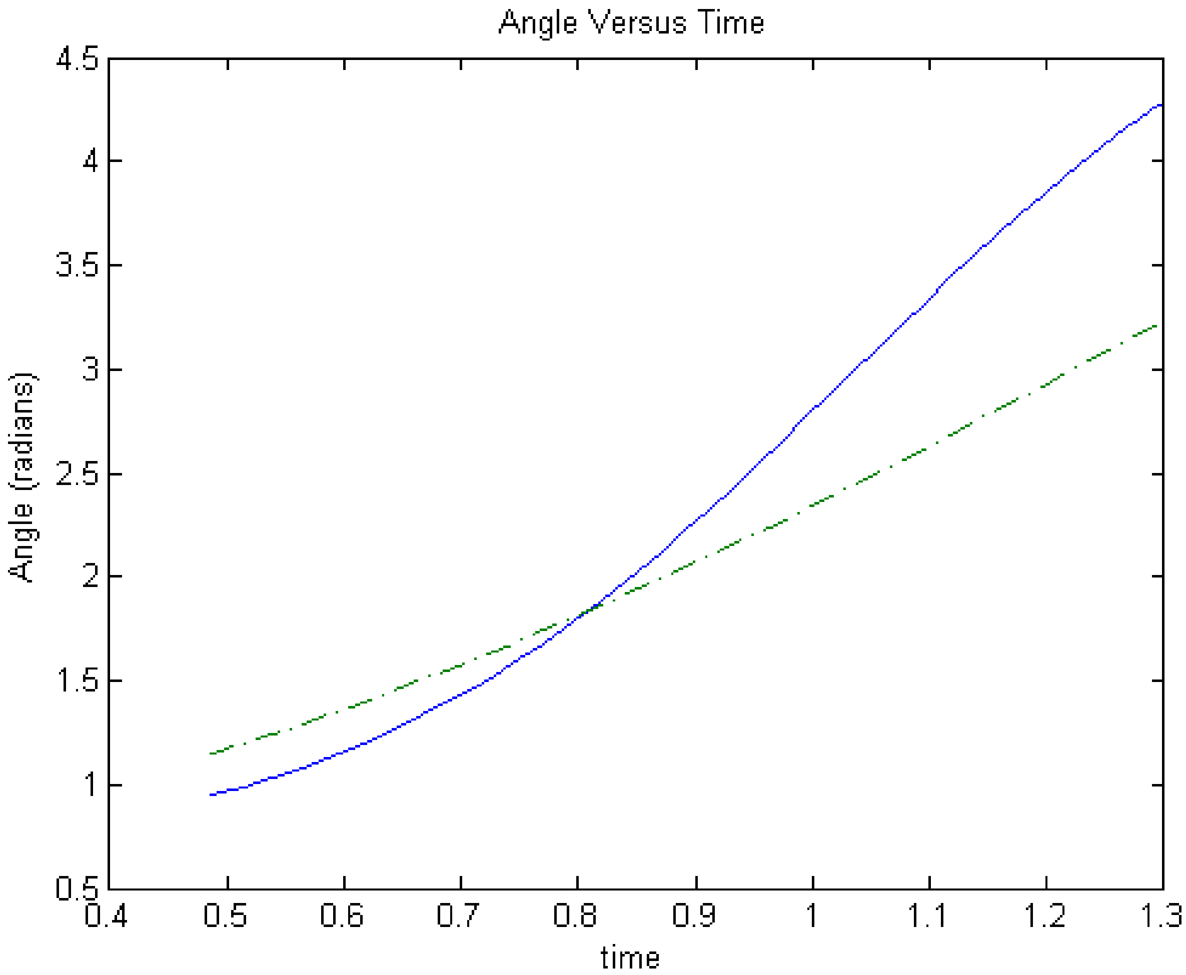
Angle Versus Time
Angle (radians)
time
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3

References

- Donald B. Siano, “Trebuchet Mechanics”, www.algobeautytreb.com/trebmath35.pdf March, 2001
- “History and Mechanics of the Trebuchet”, www.redstoneprojects.com, Google, 2009
- Anatoly Zlotnik, for MATLAB advice

Questions?