1. Show that the Laplace's Equation
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]
can be written in polar coordinates as
\[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \]

10 p/3
Calculate \( U(r, \theta) \) that satisfies the Laplace's equation in polar co-ordinates.

Assume boundary conditions given by

\[
\begin{align*}
U(1, \theta) &= 0 & & 0 \leq \theta \leq \frac{\pi}{4} \\
U(2, \theta) &= \sin \theta & & 0 \leq \theta \leq \frac{\pi}{4} \\
U(r, 0) &= 0 & & 1 \leq r \leq 2 \\
U(r, \frac{\pi}{4}) &= 0 \\
\end{align*}
\]

20 pt
Calculate $u(r, \theta)$ that satisfies the Laplace's equation in polar co-ordinates. Assume boundary conditions given by

$u(1, \theta) = 0$

$u(2, \theta) = \cos \theta.$

10 pts
Solve the 2-dimensional heat equation in polar coordinates given by

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \]

The initial condition and the boundary conditions are given above. The heat equation has to be solved over an annular region \(1 \leq |r| \leq 2\).
5. Repeat problem 4 for the region $|x| \leq 2$.

Solve the heat equation over a circular region, boundary temperature is maintained at 0.

$10/13$
Consider the 2-dimensional circular membrane equation given by

\[ \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \]

Assume circularly symmetric solution.
\[ \text{ie } \frac{\partial \psi}{\partial \theta} \equiv 0. \]

Assume boundary condition
\[ \psi(1, \theta, t) = 0 \]

and initial condition
\[ \psi(r, \theta, 0) = f(r) = 1 - r^2, \quad r \leq 1 \]
\[ \psi_t(r, \theta, 0) = g(r) = 1 - r, \quad r \leq 1 \]

Calculate \( \psi(r, \theta, t) \).