In power circuits filters are implemented with inductors and capacitors to obtain the desired filter characteristics. In integrated electronic circuits, however, it has not been possible to realize high quality inductors in integrated form, so filters are often implemented with resistors, capacitors, and amplifiers. These are called active filters. It is also possible to construct filters with just capacitors and amplifiers, so-called switched capacitor filters, but we will restrict our experiment to the conventional active filter with resistors, capacitors, and operational amplifiers.

**Network Functions**

**Low Pass Single Pole**

A low pass RC network with one capacitor has the transfer function

\[ H(s) = \frac{H_0 \omega_0}{s + \omega_0} \]  

where \( H_0 \) is the value of the transfer function at \( s = 0 \), or the dc value, \( \omega_0 \) is the natural frequency of the network, and \( s \) is the complex frequency, \( \sigma + j\omega \). For \( s = j\omega \) have

\[ \frac{H(j\omega)}{H_0} = \frac{\omega_0}{\omega_0 + j\omega} \]  

The ratio of the network transfer function to its dc value is, therefore, a complex number with a magnitude of

\[ \left| \frac{H(j\omega)}{H_0} \right| = \sqrt{\frac{\omega_0^2}{\omega_0^2 + \omega^2}} \]  

And a phase angle of

\[ \phi(\omega) = -\arctan\left(\frac{\omega}{\omega_0}\right) \]  

The group delay time, which is commonly referred to as the pulse response time, is

\[ \tau(\omega) = -\frac{d[\phi(\omega)]}{d\omega} = \frac{\cos^2 \phi(\omega)}{\omega_0} \]  

**Low Pass Double Pole**

A low pass RC network with resistors, two capacitors, and an amplifier has a transfer function with a complex conjugate pole pair given by

\[ H(s) = \frac{H_0 \omega_0^2}{s^2 + \alpha \omega_0 s + \omega_0^2} \]
Where \( 1/\alpha = Q \), the quality factor of the network. For \( s = j\omega \) the ratio of the transfer function to its dc value is

\[
\frac{H(j\omega)}{H_0} = \frac{\omega^2}{(\omega_0^2 - \omega^2) + j\omega_0\omega Q}
\]

(7)

which is a complex number whose phase angle and pulse response can be determined as in equations (4) and (5).

**Band Pass Double Pole**

A band pass two pole transfer function has the form

\[
H(s) = \frac{H_0\alpha\omega_0 s}{s^2 + \alpha\omega_0 s + \omega_0^2}
\]

(8)

The band pass function has the property that \( H(0) = H(\infty) = 0 \).

**High Pass Double Pole**

A high pass two pole transfer function has the form

\[
H(s) = \frac{H_0s^2}{s^2 + \alpha\omega_0 s + \omega_0^2}
\]

(9)

For the high pass function, \( H(0) = 0 \) while \( H(\infty) = H_0 \).

**Low Pass Double Pole Amplitude Response**

Low pass double pole functions are characterized by their natural frequency, \( \omega_0 \), and \( \alpha \), or as is more common in electronic circuit usage, their \( Q (=1/\alpha) \) or their damping ratio \( \zeta (=\alpha/2) \). Note that circuits with \( \zeta=1 \) (\( \alpha=2 \)) are “Critically Damped”. Circuits with \( \zeta<1 \) (\( \alpha<2 \)) are termed “Underdamped” while circuits with \( \zeta>1 \) (\( \alpha>2 \)) are termed “Overdamped”.

\[
H(j\omega) = \frac{\omega^2}{(\omega_0^2 - \omega^2) + j\omega_0\omega Q}
\]
Figure 1 – Amplitude response of low pass 2 pole filter with Q as parameter

Figure 1 shows the magnitude of the ratio of the transfer function of a low pass two pole filter to its dc value as a function of the ratio of the radian frequency, $\omega$, to the natural radian frequency, $\omega_0$, for several values of $Q$. At the frequency $\omega = \omega_0$, it is evident from equation (7) that this ratio is

$$\frac{H(j\omega_0)}{H_0} = -jQ = Q/90^\circ$$  \hspace{1cm} (10)

For filters with large $Q$ values the maximum gain occurs at $\omega \cong \omega_0$ and the maximum gain is $Q$. At this frequency, for example, the amplitude response of a circuit with $Q = 10$ would be, in decibels, $20 \log_{10} (10) = 20$ dB while the amplitude response of a circuit with $Q = 0.707$ would be $20 \log_{10} (0.707) = -3$ dB. More exactly, for $Q \geq 1/\sqrt{2}$, the maximum gain occurs at a frequency less than $\omega_0$, namely $\omega = \omega_0(1-1/(2Q^2))^{1/2}$ and the value of the maximum gain is $Q/(1-1/(4Q^2))^{1/2}$. Figure 2 shows a normalized amplitude response for a two-pole low pass filter. The values of the three frequencies that are of the most use in two-pole filter design are

$$\omega_1 = \omega_0 \sqrt{1 - \frac{1}{2Q^2}}$$  \hspace{1cm} (11)

$$\omega_2 = \sqrt{2} \omega_0 \sqrt{1 - \frac{1}{2Q^2}}$$  \hspace{1cm} (12)

$$\omega_3 = \omega_0 \sqrt{1 - \frac{1}{2Q^2} + \sqrt{2 - \frac{1}{Q^2} + \frac{1}{4Q^4}}}$$  \hspace{1cm} (13)
Butterworth Filter

The simplest filter optimization is the Butterworth, or maximally flat amplitude response filter. This filter has the largest value of Q that will give an amplitude response that does not rise above its dc value. Analysis shows that a low pass filter with this property would have \( Q = 0.707 \). Further, it is straightforward to show that the –3 dB bandwidth of the filter is

\[
\omega_{3\text{dB}} = \omega_0. \tag{14}
\]

Bessel Filter

The Bessel filter has the property that its group delay, as defined in equation (5), is maximally flat, in the same sense as the amplitude response is maximally flat in the Butterworth filter. Here, analysis shows that a two pole Bessel filter should have a \( Q = 0.577 \). For the Bessel two-pole low pass filter the –3 dB frequency is

\[
\omega_{3\text{dB}} = 0.785 \omega_0. \tag{15}
\]

Chebyshev Filter

All two pole low pass filters with \( Q > 0.707 \) and with no zeroes are properly called Chebyshev filters. As shown in Figure 1, they are characterized by a peak in the amplitude response. This peak is also referred to as the ‘pass band ripple’ and is usually measured in dB. For example, a 20 dB Chebyshev has a \( Q = 10 \) and maximum ripple at very close to \( \omega = \omega_0 \). The 2 dB Chebyshev shown in Figure 1 has \( Q = 1.124 \) and peaks at a frequency \( \omega = 0.777\omega_0 \). Working from equation (7) we can establish that a 2 dB two pole low pass Chebyshev filter has a –3 dB frequency such that

\[
\omega_{3\text{dB}} = 1.331 \omega_0. \tag{16}
\]
Circuit Analysis

The circuit that we will use in this experiment is shown in Figure 3. Although employing more than one amplifier, it has low sensitivity of the natural frequency and Q to circuit component value fluctuations and is capable of implementing filters with relatively large Q’s. Also, it provides at separate outputs, the low pass, band pass, and high pass transfer functions.

![Active filter circuit diagram](image)

**Figure 3 – Active filter circuit diagram.** Pin 8 is low pass output, pin 15 is band pass output, and pin 14 is high pass output.

An analysis of the circuit diagram of the active filter in Figure 3 shows that the ratio of the low pass output voltage at pin 8 to the input voltage is given by

\[
\frac{V_{\text{low}}(s)}{V_{\text{in}}} = \frac{-\frac{10^4}{R_{\text{in}} R_{F1} R_{F2} C^2}}{s^2 + \frac{R_Q}{R_Q + 10^5} \left(11R_{\text{in}} + 10^3\right) 10R_{\text{in}} R_{F1} C s + \frac{1}{10R_{F1} R_{F2} C^2}} \tag{17}
\]

Notice that equation (17) has the same form as equation (6). The band pass output at pin 15 is given by

\[
\frac{V_{\text{band}}(s)}{V_{\text{in}}} = \frac{+s\left\{\frac{10^4}{R_{\text{in}} R_{F1} C}\right\}}{s^2 + \frac{R_Q}{R_Q + 10^5} \left(11R_{\text{in}} + 10^5\right) R_{\text{in}} R_{F1} C s + \frac{1}{10R_{F1} R_{F2} C^2}} \tag{18}
\]

which has the same form as equation (8).
The high pass output at pin 14 is given by

\[
\frac{V_{\text{high}}(s)}{V_{\text{in}}} = -s^2 \left( \frac{10^4}{R_{\text{in}}} \right) \frac{\left( \frac{R_Q}{R_Q + 10^5} \right) \left( 11R_{\text{in}} + 10^5 \right)}{s^2 + \frac{10R_{\text{in}}R_{F1}C}{s + \left( \frac{1}{10R_{F1}R_{F2}C^2} \right)}}
\]

which has the same form as equation (9).

Filter Design

**Before you come to the laboratory** you will need to design a Butterworth, a Bessel, and a 2 dB Chebyshev with a –3 dB corner frequency of 5 kHz and a DC gain of 2.

Using equation (17)

\[
\frac{V_{\text{low}}(0)}{V_{\text{in}}} = -\left( \frac{10^4}{R_{\text{in}}R_{F1}R_{F2}C^2} \right) (10R_{F1}R_{F2}C^2) = -\frac{10^5}{R_{\text{in}}}
\]

Therefore, the dc gain is simply determined by \( R_{\text{in}} \). A comparison of equations (17) and (6) shows that

\[
\omega_0^2 = \frac{1}{10R_{F1}R_{F2}C^2}
\]

So the natural frequency in Hz is then given by

\[
f_0 = \frac{1}{2\pi C\sqrt{10R_{F1}R_{F2}}}
\]

Usually, \( R_{F1} \) and \( R_{F2} \) are set equal to one another in the determination of the natural frequency. Again comparing equations (17) and (6) we find that

\[
Q = \frac{\omega_0}{\left( \frac{R_Q}{R_Q + 10^5} \right) \left( 11R_{\text{in}} + 10^5 \right)} \frac{1}{10R_{\text{in}}R_{F1}C}
\]

Since the dc gain is set by \( R_{\text{in}} \) and the natural frequency is set by \( R_{F1} = R_{F2} \), the quality factor is set by \( R_Q \).

**Example**

Butterworth, Bessel, and 2 dB Chebyshev filters are required to have a dc gain of 1 and a –3 dB frequency of 1 radian/sec. From equation (14) we have for the Butterworth case that
\( \omega_0 = 1 \text{ radian/sec.} \) From equation (15) we find that for the Bessel design, \( \omega_0 = 1.274 \text{ radians/sec} \) while from equation (16) for the 2 dB Chebyshev we find that \( \omega_0 = 0.751 \text{ radians/sec}. \) Then from equation (7) corresponding phase shifts at the –3 dB frequency (= 1 radian/sec) are Butterworth: -90°, Bessel: -74.2°, and 2 dB Chebyshev: -123.1°.

**Experiment**

**Equipment List**
1. Active filter circuit printed circuit board
1. Printed circuit board fixture
1. HP 33120A Function Generator
1. HP 3580A Spectrum Analyzer

**Preliminary Preparation**

*Before you come to the laboratory* you must complete designs for 2-pole low-pass Butterworth, Bessel, and 2 dB Chebyshev filters. Design the filters to have a corner frequency of 5 kHz and a DC gain of 2. For this purpose you need to calculate values for \( R_{in}, R_{F1} = R_{F2}, \) and \( R_Q \) for each of the three filters to satisfy the design specifications.

**Procedure**

The printed circuit board with the active filter is shown in Figure 4. All components, with the exception of \( R_{F1}, R_{F2}, R_Q, \) and \( R_{in} \) are already on the board. The 2 capacitors denoted as \( C \) in Figures 3 and 4 have been made equal to 1000 pF. Input, output, and power supply connections are as shown in Figure 4. There are sockets on the board to allow insertion of the four resistor values.

![Active Filter Circuit Diagram](Figure 4 – Active filter circuit printed circuit board showing external connections)
Do not immediately connect the two dc power supplies to the circuit. First, adjust both supplies to +15 V, make the appropriate connections to obtain +15 V and -15 V with the center at ground potential, and turn the power supplies off.

1. **Butterworth Filter.** Insert the resistances you calculated for \( R_{in}, R_{F1} = R_{F2} \) and \( R_Q \) into the appropriate sockets on the printed circuit board. To obtain accurate resistance values you may need to parallel several resistors. Supply the dc bias connections indicated in Figure 22 and then turn the dc supplies on. Check to make sure you still have +15 and -15 V.

   a) With a sinusoidal input sufficient to give about 5 V peak to peak output, measure the magnitude and phase of the low-pass voltage transfer ratio, \( V_S/V_{in} \), from about 1 Hz to 50 kHz paying particular attention to the frequency range around cut off. Beyond the cut off frequency you may want to increase the amplitude of the input signal to obtain more reliable readings.

   b) With a 30 Hz square wave input sufficient to give about 10 V peak to peak output, adjust the oscilloscope to trigger on the rising edge of the input pulse, and record the waveform of the low-pass output at pin 8. This signal is a characterization of the step response of the filter.

   c) Connect the tracking oscillator output of the HP 3580A spectrum analyzer to the input of the filter. Record as dimensioned sketches the Bode plots for the low-pass output at pin 8, the band-pass output at pin 15, and the high-pass output at pin 14. Be sure to record the maximum gain amplitudes for the band-pass output and the high-pass output as compared to the maximum low-pass output. It will be necessary to establish a reference level on the spectrum analyzer by connecting the tracking oscillator output to the input of the spectrum analyzer.

2. **Bessel Filter.** With the dc power supplies off, insert the resistances you calculated for \( R_{in}, R_{F1} = R_{F2} \) and \( R_Q \). Now turn the dc supplies on. Check to make sure you still have +15 and -15 V.

   a) With a sinusoidal input sufficient to give about 5 V peak to peak output, measure the magnitude and phase of the low-pass voltage transfer ratio, \( V_S/V_{in} \), from about 1 Hz to 50 kHz paying particular attention to the frequency range where the amplitude change is greatest.

   b) With a 30 Hz square wave input sufficient to give about 10 V peak to peak output, record the waveform of the low-pass output at pin 8.

3. **Chebyshev Filter.** With the dc power supplies off, insert the resistances you calculated for \( R_{in}, R_{F1} = R_{F2} \) and \( R_Q \). Now turn the dc supplies on. Check to make sure you still have +15 and -15 V.

   a) With a sinusoidal input sufficient to give about 5 V peak to peak output, measure the magnitude and phase of the low-pass voltage transfer ratio, \( V_S/V_{in} \), from about 1 Hz to 50 kHz paying particular attention to the frequency range where the output amplitude changes rapidly. Here it may be useful to look at a Bode plot first.

   b) With a 30 Hz square wave input sufficient to give about 10 V peak to peak output, record the waveform of the low-pass output at pin 8.
Report

(a) Tabulate your calculated values of $R_{in}$, $R_{F1} = R_{F2}$, and $R_Q$ for the 2-pole low-pass Butterworth, Bessel, and Chebyshev filters. Show your calculations in detail. Describe any differences between these values and the values actually used in the experiment.

(b) On the same graph plot the low-pass voltage transfer ratio versus frequency for all three low-pass filters for comparison. Use a log scale for frequency. On another graph plot the phase shift versus frequency for all three low-pass filters. Discuss the differences in these results for the three filters.

(c) Tabulate the measured values of –3 dB frequency, the phase shift at the –3 dB frequency, and the frequency at –90° phase shift for all three low-pass filters. Discuss how these values compare with the design values.

(d) Present and discuss differences in the step response for the three low-pass filters paying particular attention to any ringing.

(e) On the same graph present the Bode plots for the low-pass, band-pass, and high-pass outputs for the Butterworth filter. Compare the experimental maximum gains for the low-pass, band-pass, and high-pass outputs to those expected theoretically. Quantitative, not qualitative, results are expected here. Show your calculations in detail.

References