Lec 1

($\mathbb{R}^3$ Simplified)

1.1 A vector quantity $\mathbf{u}$ is an ordered number triple $(x, y, z)$ in which $x, y, z$ are real numbers. We write $\mathbf{u} = (x, y, z)$.

$x, y$ and $z$ are the 1st, 2nd & 3rd components of $\mathbf{u}$.

Fact: $(0, 0, 0)$ is called the null vector $\mathbf{0} = (0, 0, 0)$

1.2 Norm of a vector:

$$||\mathbf{u}|| = \sqrt{x^2 + y^2 + z^2}$$

$||\mathbf{u}||$ is called the 'magnitude' or length of $\mathbf{u}$. 

\[ \| 0 \| = 0 \]

**Fact:** If \( \| u \| = 1 \), then \( u \) is called an **unit vector**.

**Ex:** \(( \cos \theta \ \sin \theta \ 0 )\)

is an **unit vector** for **any** \( \theta \).

**Ex** \(( \cos \theta \ \cos \phi \ \cos \theta \sin \phi \ \sin \theta )\)

is an **unit vector** for **any** \( \theta \) and \( \phi \).

### 1.3 Sum of Two Vectors:

\[
\begin{align*}
\mathbf{u} &= (x_1 \ y_1 \ z_1) \\
\mathbf{v} &= (x_2 \ y_2 \ z_2) \\
\mathbf{u} + \mathbf{v} &= (x_1 + x_2 \ y_1 + y_2 \ z_1 + z_2)
\end{align*}
\]

**Ex:** \( \mathbf{u} = (1 \ 2 \ -5) \)

\( \mathbf{v} = (-2 \ 2 \ 4) \)

\( \mathbf{u} + \mathbf{v} = (1-2 \ 2+2 \ -5+4) \)

\( = (-1 \ 4 \ -1) \)
1.4 Scalar Multiplication:

\[ \mathbf{u} = (x \ y \ z) \]

\( \lambda \) is a scalar (real number).

\[ \lambda \mathbf{u} = (\lambda x \ \lambda y \ \lambda z) \]

Ex: \( \mathbf{u} = (1 \ 2 \ -5) \)

\[ 3 \mathbf{u} = (3 \cdot 1 \ 3 \cdot 2 \ 3 \cdot (-5)) \]

\[ = (3 \ 6 \ -15) \]

\[ \mathbf{v} = (-2 \ 2 \ 4) \]

\[ 4 \mathbf{v} = (4 \cdot (-2) \ 4 \cdot 2 \ 4 \cdot 4) \]

\[ = (-8 \ 8 \ 16) \]

\[ 3 \mathbf{u} + 4 \mathbf{v} = (3 - 8 \ 6 + 8 \ -15 + 16) \]

\[ = (-5 \ 14 \ 1) \]
Fact:
\[ \|\lambda u\| = |\lambda| \|u\| \]

1.5. Vector addition and triangle inequality

1. \[ \|u + v\| \leq \|u\| + \|v\| \]
2. \[ \|u - v\| \geq \|\|u\| - \|v\|\| \]

1.6. The \( i, j \) and \( k \):
\[
i = (1 \ 0 \ 0) \\
j = (0 \ 1 \ 0) \\
k = (0 \ 0 \ 1)
\]

\[
u = (x \ y \ z) = x(1 \ 0 \ 0) + y(0 \ 1 \ 0) + z(0 \ 0 \ 1) = xi + yj + zk
\]
1.7 Simple geometrical application.

Midpoint: \((x_1, y_1, z_1)\)

\[C = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)\]

Straight line

\[u = (x_1, y_1, z_1)\]

\[v = (x_2, y_2, z_2)\]

\[u + \lambda v = \begin{pmatrix} x_1 + \lambda x_2 \\ y_1 + \lambda y_2 \\ z_1 + \lambda z_2 \end{pmatrix}\]

Vector Eqn of line \(l\)
If \((x, y, z)\) is a point on \(l\) we have:

\[
\begin{align*}
  x &= x_1 + \lambda x_2 \\
  y &= y_1 + \lambda y_2 \\
  z &= z_1 + \lambda z_2
\end{align*}
\]

\[
\frac{x-x_1}{x_2} = \frac{y-y_1}{y_2} = \frac{z-z_1}{z_2}
\]

Equation of a straight line

**Ex:** \[
\frac{x-5}{4} = \frac{y+2}{3} = -\frac{z-7}{8}
\]

\((5, -2, 7)\) is any point on the line

\((4, 3, -8)\) is a vector parallel to the line.
Ex

\[
\frac{2x - 3}{8} = \frac{3y + 4}{9} = \frac{3 - 18}{20}
\]

Write it as

\[
\frac{x - \frac{3}{2}}{4} = \frac{y + \frac{4}{3}}{3} = \frac{3 - 18}{20}
\]

\[
\left(\frac{3}{2}, -\frac{4}{3}, 18\right) \text{ is a point on the line}
\]

\[
(4, 3, 20) \text{ is a vector parallel to line } l.
\]

Note that these vectors are not unique. Can you find another vector pair for \( l \).
Ex \[ \frac{3x-1}{4} = \frac{2y+3}{2} = \frac{2-33}{1} \]

(a) Find a unit vector \( \mathbf{u} \) parallel to \( \mathbf{l} \).

(b) Find the vector \( \mathbf{v} \) on \( \mathbf{l} \) which is perpendicular to \( \mathbf{u} \).

(c) Are the vectors \( \mathbf{u}, \mathbf{v} \) unique??

Ex

Calculate the \( d \) median vector \( \mathbf{P} \).
1.8 The Dot Product (Scalar product)

\[ U \cdot V = ||U|| \cdot ||V|| \cdot \cos \theta \] \( \star \)

Fact:
- Dot product is commutative
- Dot product is distributive and linear

\[ U \cdot V = V \cdot U \]
\[ U \cdot (V + W) = U \cdot V + U \cdot W \]
\[ U \cdot (\alpha V + \beta W) = \alpha U \cdot V + \beta U \cdot W \]

Note that \( \star \) is not a very useful definition because we don't know what \( \theta \) is.

We write
\[ U = (x_1, y_1, z_1) = x_1 i + y_1 j + z_1 k \]
\[ V = (x_2, y_2, z_2) = x_2 i + y_2 j + z_2 k \]
\[
\begin{align*}
\text{i} & = (1, 0, 0) \\
\text{i} \cdot \text{i} & = ||\text{i}|| \ ||\text{i}|| \cos \theta = 0 \\
& = 1 \\
\text{j} \cdot \text{j} & = 1 \\
\text{k} \cdot \text{k} & = 1 \\
\text{i} \cdot \text{j} & = ||\text{i}|| \ ||\text{j}|| \cos 90^\circ = 0 \\
\text{etc.}
\end{align*}
\]

It follows that

\[
\text{u} \cdot \text{v} = x_1x_2 + y_1y_2 + z_1z_2
\]

This is useful.

\[
\cos \theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2} \sqrt{x_2^2 + y_2^2 + z_2^2}}
\]

\[0 \leq \theta \leq \pi\]
1.9 \[ u = (x_1, y_1, z_1) \]
\[ u \cdot u = x_1^2 + y_1^2 + z_1^2 \]
\[ = \| u \|^2 \]

Hence \[ \| u \| = \sqrt{u \cdot u} \].

If \( u \cdot v = 0 \) then we say that \( u \) and \( v \) are orthogonal or perpendicular.

**Ex:** \( u = (1, 2, 3) \)
\[ v = (2, -1, -2) \]
\[ \| u \| = \sqrt{14} \quad \| v \| = 3 \quad u \cdot v = -6 \]
\[ \cos \theta = \frac{-6}{\sqrt{14} \cdot 3} = -\frac{2}{\sqrt{14}} \]
\[ \theta = 122.3^\circ \]
Ex: Find the strength of the magnetic field vector \( \mathbf{H} = (5 \ 3 \ 7) \) in the direction \( (2 \ -1 \ 2) \).

We need to project \( \mathbf{H} \) in the direction of \( \mathbf{u} \).

\[
\lVert \mathbf{u} \rVert = 3
\]

Define \( \hat{\mathbf{u}} = \frac{\mathbf{u}}{\lVert \mathbf{u} \rVert} = \left( \frac{2}{3} \ - \frac{1}{3} \frac{2}{3} \right) \)

\( \hat{\mathbf{u}} \) is a unit vector in the direction \( \mathbf{u} \).

Strength of \( \mathbf{H} \) in the direction of \( \mathbf{u} \) is

\[
H \cdot \hat{\mathbf{u}} = \lVert \mathbf{H} \rVert \cos \theta = \frac{10}{3} - \frac{3}{3} + \frac{14}{3} = 7
\]
1.10: Direction cosines -

\[ \mathbf{u} = (a, b, c) \]

We want to calculate the angle \( \theta_x \) between \( \mathbf{u} \) and the \( x \) axis.

\[ \mathbf{u} \cdot \mathbf{i} = \cos \theta_x \cdot ||\mathbf{u}|| \cdot ||\mathbf{i}|| \]

But \( \mathbf{u} \cdot \mathbf{i} = a \)

Hence

\[ \cos \theta_x = \frac{a}{\sqrt{a^2 + b^2 + c^2}} \]

Define \( l = \cos \theta_x \) called the direction cosine of the angle between \( \mathbf{u} \) and the \( x \) axis.

Likewise we have

\[ m = \cos \theta_y = \frac{b}{\sqrt{a^2 + b^2 + c^2}} \]
\[ n = \cos \theta_3 = \frac{c}{\sqrt{a^2 + b^2 + c^2}} \]

1, m, n are the three direction cosines.

Note that \( l^2 + m^2 + n^2 = 1 \).

Note also that
\[ u = \|u\| (l \ m \ n) \]
\[ \text{or } \|u\| (l \ i + m \ j + n \ k) \]
1.11: Cauchy-Schwarz Inequality

\[ |u \cdot v| \leq \|u\| \|v\| . \]

An interesting calculation.

\[ \|u + \lambda v\|^2 = (u + \lambda v) \cdot (u + \lambda v) \]

\[ = u \cdot u + \lambda^2 v \cdot v + u \cdot \lambda v + \lambda v \cdot u \]

\[ = \|u\|^2 + \lambda^2 \|v\|^2 + 2 \lambda u \cdot v . \]

Since the above expression is true for any \( \lambda \) it follows that in particular it is true for

\[ \lambda = -\frac{\|u\|^2}{u \cdot v} \quad \text{This is a trick}. \]
Thus,

\[ \|u + \alpha v\|^2 = \]

\[ \|u\|^2 + \frac{\|u\|^4}{(u \cdot v)^2} \|v\|^2 + \]

\[ 2 \left( -\frac{\|u\|^2}{u \cdot v} \right)(u \cdot v) \]

\[ = -\|u\|^2 + \frac{\|u\|^4 \|v\|^2}{(u \cdot v)^2} \]

Since the l.h.s is always \( \geq 0 \)
we have

\[ \frac{\|u\|^4 \|v\|^2}{(u \cdot v)^2} \geq \|u\|^2 \]

Hence \((u \cdot v)^2 \leq \|u\|^2 \|v\|^2 \)
Assuming \( \|u\| \neq 0 \).
and \( |u \cdot v| \leq \|u\| \|v\| \).
If we set $\lambda = 1$ in the expression

$$\|u + \lambda v\|^2 = \|u\|^2 + \lambda^2 \|v\|^2 + 2\lambda u \cdot v,$$

we obtain

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2u \cdot v.$$

$$\leq \|u\|^2 + \|v\|^2 + 2\|u\| \|v\|$$

Hence

$$\|u + v\|^2 \leq \left[\|u\| + \|v\|\right]^2$$

or

$$\|u + v\| \leq \|u\| + \|v\|$$

(Triangle Inequality).
1.12 Equation of a plane

\[ \mathbf{n} \cdot (\mathbf{r} - \mathbf{a}) = 0 \]

- \( P \) is a plane
- \( \mathbf{a} \) is a vector on the plane
- \( \mathbf{n} \) is a vector perpendicular to the plane
- \( \mathbf{r} \) is any other point on the plane
\( \mathbf{a} = (a_1, a_2, a_3) \)

\( \mathbf{n} = (n_1, n_2, n_3) \)

\( \mathbf{r} = (x, y, z) \)

we have

\[
\mathbf{n} \cdot (\mathbf{r} - \mathbf{a}) = 0 \quad \text{Vector equation of the plane.}
\]

\[
(n_1, n_2, n_3) \cdot (x - a_1, y - a_2, z - a_3) = 0
\]

\[
\begin{align*}
(n_1 x + n_2 y + n_3 z) &= n_1 a_1 + n_2 a_2 + n_3 a_3 \\
\end{align*}
\]

\(\uparrow \) "Cartesian Eqn of the Plane"
Ex: Consider a plane $P$ through the point $(2, 5, 3)$ with normal $(3, 2, -7)$.

$$n = (3, 2, -7)$$

$$3x + 2y - 7z = 6 + 10 - 21 = -5$$

$$3x + 2y - 7z = -5$$
1.13 Vector description of a plane

\[ a = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{8}{7} \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \]

are 3 points on the plane \( P \).

Define

\[ u = b_1 - a = \begin{pmatrix} -1 \\ -5 \\ -\frac{13}{7} \end{pmatrix}, \]

\[ v = b_2 - a = \begin{pmatrix} -2 \\ -4 \\ -2 \end{pmatrix}. \]
P is given by the set of points

\[
\begin{pmatrix}
2 \\
5 \\
3
\end{pmatrix} + \lambda_1 \begin{pmatrix}
-1 \\
-5 \\
-13/7
\end{pmatrix} + \lambda_2 \begin{pmatrix}
-2 \\
-4 \\
-2
\end{pmatrix}
\]

for \( \lambda_1, \lambda_2 \in \mathbb{R} \).

(compare this with page 5)

\[ a + \lambda_1 (b_1 - a) + \lambda_2 (b - a) \]

is the vector describing the plane.