ESE 501, Home Work 6

1. Assuming that $\lambda_0$ is any nonzero real number, compute exponential of the following matrices.
   (a)\[ \begin{pmatrix} \lambda_0 & 1 & 0 & 0 \\ 0 & \lambda_0 & 1 & 0 \\ 0 & 0 & \lambda_0 & 1 \\ 0 & 0 & 0 & \lambda_0 \end{pmatrix} \] (1)
   (b)\[ \begin{pmatrix} \lambda_0 & 0 & 1 & 0 \\ 0 & \lambda_0 & 0 & 1 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_0 \end{pmatrix} \] (2)
   (c)\[ \begin{pmatrix} \lambda_0 & 0 & 0 & 1 \\ 0 & \lambda_0 & 0 & 0 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_0 \end{pmatrix} \] (3)
   (d)\[ \begin{pmatrix} \lambda_0 & 0 & 1 & 1 \\ 0 & \lambda_0 & 0 & 1 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_0 \end{pmatrix} \] (4)
   (e)\[ \begin{pmatrix} \lambda_0 & 1 & 0 & 0 \\ 1 & \lambda_0 & 1 & 0 \\ 0 & 1 & \lambda_0 & 1 \\ 0 & 0 & 1 & \lambda_0 \end{pmatrix} \] (5)

Remark: I have never tried (d) and (e). You can use any method except typing the matrix into a symbolic computer.

2. (a) Calculate $e^A$ and $e^{A^2}$ assuming $A$ to be the following matrix:
    \[ A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix} \] (6)

(b) Verify, using matlab if necessary, that $A, e^A$ and $e^{A^2}$ commute.

Remark: I am expecting that you will compute the eigenvalues and compute the exponential function by expressing it as a linear combination of $I, A$ and $A^2$. You can use matlab to compute the linear combination. Do not use the exponential command.

3. (a) Calculate eigenvalues, eigenvectors and generalized eigenvectors of the following matrix.
    \[ M = \begin{pmatrix} 45 & 120 & 315 & 672 \\ -60 & -252 & -756 & -1680 \\ 45 & 216 & 651 & 1440 \\ -12 & -60 & -180 & -396 \end{pmatrix} \] (7)

Remark: You can use matlab and if you do, you need to verify your answer by using the definition of eigenvalues, eigenvectors and generalized eigenvectors.
(b) Write $e^{Mt}$ as a linear combination of $I, M, M^2$ and $M^3$ using the eigenvalues of $M$.

(c) Using the matlab ‘jordan’ command, obtain the matrix $P$ such that $P^{-1}MP$ is block diagonal. Verify your results by actually multiplying the matrices using matlab.

4. Show that the matrix

$$M = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-2 & 5 & 0 & -3 & 10
\end{pmatrix}$$

has an eigenvector at $(1, \lambda, \lambda^2, \lambda^3)$ for every eigenvalue $\lambda$. Also show that the eigenvalues are at 1, 2, 3 and 4. Write down the matrix $P$ (without using matlab) such that $P^{-1}MP$ is diagonal. Now use matlab to verify that $P^{-1}MP$ is indeed diagonal.

5. Manufacture a $10 \times 10$ symmetric and positive definite matrix $M$. Calculate eigenvalues of the matrix $M$. Also calculate an orthogonal matrix $P$ such that $P^T MP$ is diagonal. Verify by explicit multiplication that $PP^T = P^T P = I$ (You can use matlab for this purpose).

6. Let $A$ be a $n \times n$ matrix. Also assume that the eigenvalues of $A$ are all distinct and let $v_i$ be an eigenvector of $A$ corresponding to an eigenvalue $\lambda_i$. Prove that the vectors $v_i, i = 1, \cdots n$ are all linearly independent.
Problem 7:
1) Let $A$ be an $n \times n$ matrix. Assume that there exists a nonsingular matrix $P$ such that $P^{-1}AP = B$. Show that if $v$ is an eigenvector of $B$ for an eigenvalue $\lambda$ then $Pv$ is an eigenvector of $A$ for the same eigenvalue $\lambda$.
2) Assume that $A$ is a $3 \times 3$ matrix with characteristic polynomial given by
   \[ p(\lambda) = \lambda^3 - c_3 \lambda^2 - c_2 \lambda - c_1. \]
   Let $b$ be a $3 \times 1$ vector such that $b, Ab$ and $A^2b$ are linearly independent, show that $v_3 = b, v_2 = Ab - c_3b$, $v_1 = A^2b - c_3Ab - c_2b$ are also linearly independent. Define a matrix $P$ as
   \[ P = [v_1, v_2, v_3]. \]
   Verify that
   \[ P^{-1}AP = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_1 & c_2 & c_3 \end{pmatrix}. \]
   If $\lambda_1$ is an eigenvalue of the matrix $A$, show that
   \[ v_1 + v_2 \lambda_1 + v_3 \lambda_1^2 \]
   is the corresponding eigenvector.
3) Assume that $A$ is a $5 \times 5$ matrix with characteristic polynomial given by
   \[ p(\lambda) = \lambda^5 - c_5 \lambda^4 - c_4 \lambda^3 - c_3 \lambda^2 - c_2 \lambda - c_1. \]
   Let $b$ be a $5 \times 1$ vector such that $b, Ab, A^2b, A^3b$ and $A^4b$ are linearly independent. Can you write down a matrix $P$ such that
   \[ P^{-1}AP = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{pmatrix}. \]
   If $\lambda_1$ is an eigenvalue of the matrix $A$, what is the corresponding eigenvector?
Problem 8: Let $A$ be a $17 \times 17$ matrix and assume that there exists a $17 \times 17$ invertible matrix $P$ such that

$$P^{-1} A P = \Sigma,$$

where

$$\Sigma = \begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 & 0 \\
0 & \lambda & 1 & \cdots & 0 & 0 \\
0 & 0 & \lambda & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & \lambda \\
\end{pmatrix}.$$

1) Calculate $\Sigma^N$ for some integer $N \geq 17$.
2) Calculate $A^N$.

Hint: Write $\Sigma = \Lambda + \Xi$, where

$$\Lambda = \begin{pmatrix}
\lambda & 0 & 0 & \cdots & 0 & 0 \\
0 & \lambda & 0 & \cdots & 0 & 0 \\
0 & 0 & \lambda & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & \lambda \\
\end{pmatrix}$$

and

$$\Xi = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.$$

- Show that $\Lambda$ and $\Xi$ commute i.e. $\Lambda \Xi = \Xi \Lambda$.
- Calculate $\Xi^j$, for $j = 1, \cdots, 16$. Also show that $\Xi^j = 0$, for $j \geq 17$. (Such a matrix $\Xi$ is called a Nilpotent matrix.)
- Using Binomial expansion, write down

$$\Sigma^N = \Lambda^N + N \binom{N}{1} \Lambda^{N-1} \Xi + N \binom{N}{2} \Lambda^{N-2} \Xi^2 + \cdots + N \binom{N}{16} \Lambda^{N-16} \Xi^{16},$$

where

$$N \binom{N}{m} = \frac{N!}{m! (N-m)!}.$$ 

- Finally write down

$$\Sigma^N = \begin{pmatrix}
\lambda^N & N \binom{N}{1} \lambda^{N-1} \lambda^N & N \binom{N}{2} \lambda^{N-2} \lambda^N & \cdots & N \binom{N}{15} \lambda^{N-15} \lambda^N & N \binom{N}{16} \lambda^{N-16} \\
0 & \lambda^N & N \binom{N}{1} \lambda^{N-1} \lambda^N & \cdots & N \binom{N}{14} \lambda^{N-14} \lambda^N & N \binom{N}{15} \lambda^{N-15} \\
0 & 0 & \lambda^N & \cdots & N \binom{N}{14} \lambda^{N-14} \lambda^N & \cdots \\
0 & 0 & 0 & \lambda^N & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & \lambda^N \\
\end{pmatrix},$$

and show that

$$A^N = P \Sigma^N P^{-1}.$$

Remark: Hopefully this exercise illustrates the power of jordan form, similarity transform and nilpotent matrices.