Homework Nine

1. Let
   \[ T : \mathbb{R}^3 \to \mathbb{R}^4 \]
   be a linear transformation defined as follows
   \[
   \begin{pmatrix}
   x_1 \\
   x_2 \\
   x_3
   \end{pmatrix}
   \mapsto
   \begin{pmatrix}
   x_1 + x_2 \\
   x_2 + x_3 \\
   x_3 - x_1 \\
   2x_1 + x_2 - x_3
   \end{pmatrix}
   \]
   (a) Write down the range and null space of \( T \) as span of a set of linear independent vectors.
   (b) Obtain the cartesian equation of the range and null space.
   (c) Choose a point \( p = (3, 5, 2, 1)^T \) in the range of \( T \). Describe the pre image of \( T \), i.e. find
   \[ S = \{ x \in \mathbb{R}^3 : T(x) = p \}. \]
   (d) Is the space \( S \) (in part (c)) a vector space? If not, what is it?
   (e) Let \( P \) be a plane in \( \mathbb{R}^3 \) described by the equation
   \[ x_1 - x_3 = 0. \]
   i. Calculate the image of \( P \) under the transformation \( T \), call it \( T(P) \).
   ii. Is \( T(P) \) a vector space, justify with a reason. If not, what is it?

2. Let
   \[ A = \begin{pmatrix}
   2 & 5 & 9 \\
   3 & 2 & 6 \\
   1 & -3 & 8
   \end{pmatrix}, \]
   and let \( b \) be a \( 3 \times 1 \) vector with the property that \( b, Ab, A^2b \) are linearly independent, so that
   \[ B = \{ b, Ab, A^2b \} \]
   form a basis of \( \mathbb{R}^3 \). Define a linear transformation
   \[ T : \mathbb{R}^3 \to \mathbb{R}^3 \]
   where
   \[ A^{n-1}b \mapsto A^n b, n = 1, 2, 3, 4, \ldots \]
   (a) If a vector \( v \) has coordinates \( (1, 2, 5) \) with respect to basis \( B \), i.e.
   \[ v = b + 2Ab + 5A^2b \]
   find coordinates of \( T(v) \) with respect to the same basis \( B \).
   (b) If a vector \( v \) has coordinates \( (\alpha, \beta, \gamma) \) with respect to basis \( B \), i.e.
   \[ v = \alpha b + \beta Ab + \gamma A^2b \]
   find coordinates of \( T(v) \) with respect to the same basis \( B \).

3. Let \( P_3(t) \) be the set of all polynomials in \( t \) of degree \( \leq 3 \). Define a linear transformation
   \[ T : P_3(t) \to P_3(t) \]
   where
   \[ p(t) \mapsto \frac{d}{dt} p(t). \]
(a) Calculate the range and null space of $T$.

(b) Let
\[ B = \{ 1, 1 + t, t - t^2, t^3 \} \]
be a basis of $P_3(t)$. If a polynomial $p(t)$ has coordinates $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ with respect to the basis $B$. Calculate the coordinates $(\alpha_2, \beta_2, \gamma_2, \delta_2)$ of $T(p(t))$ with respect to $B$.

(c) Write down a matrix $M$ such that
\[
M \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \\ \delta_2 \end{pmatrix}
\]
for all possible choices of coordinates $\alpha_1, \beta_1, \gamma_1$ and $\delta_1$.

4. Let $P_3(t)$ be the set of all polynomials in $t$ of degree $\leq 3$, and let $P_2(t)$ be the set of all polynomials in $t$ of degree $\leq 2$. Define a linear transformation
\[ T : P_2(t) \to P_3(t) \]
where
\[ p(t) \rightarrow \int p(t) \, dt. \]

(a) Calculate the range and null space of $T$.

(b) Let
\[ B_1 = \{ 1, 1 + t, t - t^2, t^3 \} \]
be a basis of $P_3(t)$ and let
\[ B_2 = \{ 1, 1 + t, t - t^2, t^3 \} \]
be a basis of $P_2(t)$. If a polynomial $p(t)$ has coordinates $(\alpha_1, \beta_1, \gamma_1)$ with respect to the basis $B_2$. Calculate the coordinates $(\alpha_2, \beta_2, \gamma_2, \delta_2)$ of $T(p(t))$ with respect to $B_1$.

(c) Write down a matrix $M$ such that
\[
M \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix}
\]
for all possible choices of coordinates $\alpha_1, \beta_1$ and $\gamma_1$.

5. Consider the ordinary differential equation
\[ \dot{x} = Ax + bu, \quad y = c^T x \]
where $x(0) = 0$, $c = (1 \ 0)$,
\[ A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \]
and
\[ b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

The solution $y(t)$ can be written as
\[ y(t) = \int_0^t e^{A(t-\tau)} b u(\tau) d\tau. \]
(a) Assume $u(t) = \alpha + \beta t$ for $\alpha, \beta \in \mathbb{R}$, calculate $y(t)$ in terms of $\alpha$ and $\beta$ and show that it is a polynomial of degree $\leq 3$, i.e.,

$$y(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3.$$ 

(b) Define a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$ as follows

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$$

where

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$  

Calculate the range and the null space of $T$.

6. Let us define

$$A = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

Recall that an ordinary differential equation

$$\dot{x} = Ax + bu, \quad x(0) = 0$$

has the following solution

$$x(t) = \int_0^t e^{A(t-\tau)}bu(\tau)d\tau.$$  

At $t = 1$, it follows that

$$x(1) = \int_0^1 e^{A(1-\tau)}bu(\tau)d\tau,$$

where $x(1) \in \mathbb{R}^2$, and $u(\tau)$ is a function defined in the interval $[0, 1]$.

(a) Calculate $x(1)$ for $u(\tau) = 1$, $\tau \in [0, 1]$.

(b) Calculate $x(1)$ for $u(\tau) = \tau$, $\tau \in [0, 1]$.

(c) Let $c_1$ and $c_2$ be two arbitrary real constants, we define

$$u(\tau) = b^T e^{-A^T \tau} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

and write $x(1)$ as

$$x(1) = e^A \left[ \int_0^1 e^{-A^T \tau} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} d\tau \right].$$

i. Calculate the $2 \times 2$ matrix

$$M = \left[ \int_0^1 e^{-A^T \tau} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} d\tau \right]$$

and check its rank. Is it 2?

ii. Calculate the $2 \times 2$ matrix

$$M = \left[ \int_0^1 e^{-A^T \tau} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} d\tau \right]$$

taking the matrices $A$ and $b$ as

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and check its rank. Is it 2 or a 1?
iii. Choose
\[ x(1) = \begin{pmatrix} 7 \\ 9 \end{pmatrix} \]

and taking \( M \) as in part (i), calculate
\[ \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = M^{-1} e^{-At} x(1) \]

and hence calculate
\[ u(\tau) = y^T e^{-AT \tau} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} . \]

Remark: the constructed \( u(\tau) \) is the forcing function (also called the control) that drives the state from \( x(0) = (0 \quad 0)^T \) to \( x(1) = (7 \quad 9)^T \).

7. Let us define
\[ A = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix}, \quad c^T = (1 \quad 0), \]

and consider an ordinary differential equation of the form
\[ \dot{x} = Ax, \quad y = c^T x, \quad x(0) = x_0. \]

Recall that \( y(t) \) is given as follows
\[ y(t) = c^T e^{At} x_0. \]

(a) Calculate \( y(t) \) assuming
\[ x_0 = \begin{pmatrix} 5 \\ 7 \end{pmatrix} . \]

(b) We would now like to calculate \( x_0 \) given \( y(t) \). This is done as follows:
\[ y(t) = c^T e^{At} x_0 \]
implies that
\[ cy(t) = c c^T e^{At} x_0 \]
which further implies that
\[ e^{AT} cy(t) = e^{AT} c c^T e^{At} x_0. \]

Integrating both sides with respect to \( t \) in the interval \([0 \quad 1]\) we obtain
\[ \int_0^1 e^{AT} cy(t) dt = \left[ \int_0^1 e^{AT} c c^T e^{At} dt \right] x_0. \]

i. Calculate the \( 2 \times 2 \) matrix \( N \) given by
\[ N = \left[ \int_0^1 e^{AT} c c^T e^{At} dt \right] \]
and check its rank. Is it \( 2 \)?

ii. Choose \( y(t) \) in part (a) and calculate
\[ \xi = \int_0^1 e^{AT} cy(t) dt. \]

iii. Writing \( \xi = N x_0 \), calculate \( x_0 = N^{-1} \xi \).

The matrices \( M \) and \( N \) in the problems 6 and 7 are called ‘Controllability’ and ‘Observability’ Gramians respectively.