1. Verify by explicit differentiation that

\[ x(t) = \int_0^t e^{A(t-\tau)} b \ u(\tau) \ d\tau \]

would satisfy the ordinary differential equation

\[ \dot{x}(t) = A \ x(t) + b \ u(t), \]

where \( x(0) = 0. \)

2. Let \( A \) be a \( n \times n \) matrix with eigenvalues at \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n. \) Show that

\[ det \ A = \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n, \]

and

\[ trace \ A = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n. \]

3. Let \( A \) be a \( 3 \times 3 \) matrix with distinct eigenvalues at \( \lambda_1, \lambda_2, \lambda_3. \) Define a \( 6 \times 6 \) matrix \( B \) as follows

\[ B = \begin{pmatrix} A & I \\ 0 & A \end{pmatrix}. \]

(a) If \( v_1, v_2 \) and \( v_3 \) are three linearly independent eigenvectors of \( A, \) show that

\[ \begin{pmatrix} v_1 \\ 0 \end{pmatrix}, \begin{pmatrix} v_2 \\ 0 \end{pmatrix}, \begin{pmatrix} v_3 \\ 0 \end{pmatrix}, \]

are three linearly independent eigenvectors of \( B. \)

(b) Show that \( B \) does not have any other linearly independent eigenvectors. In fact the three other linearly independent generalized eigenvectors are

\[ \begin{pmatrix} v_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ v_2 \end{pmatrix}, \begin{pmatrix} v_3 \\ v_3 \end{pmatrix}, \]

4. A \( 3 \times 3 \) matrix \( A \) has eigenvalues repeated at \(-2, -2 \) and \(-2\) and a single chain of generalized eigenvectors at

\[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \]

(a) Calculate \( e^{At} \) from this data.

(b) Can you write down the matrix \( A \) from this data?

5. Let \( A \) be a \( 2 \times 2 \) matrix with eigenvalues repeated at \( 0.3, 0.3. \) Calculate

\[ \sum_{j=1}^{N} j \ A^{j-1}, \text{ and } \sum_{j=1}^{\infty} j \ A^{j-1}, \]

in terms of \( A. \)