Hint for H. W. 6

1. \( d \)

\[
\begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = 0
\]

This is a nilpotent matrix.

1. \( e \)

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

is not nilpotent

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

I would use Matlab to compute its exponential.
(2) ⑤
If $A$ has eigenvalues at $\lambda_1, \lambda_2, \lambda_3$

$A^2 \quad \text{at} \quad \lambda_1^2, \lambda_2^2, \lambda_3^2$

(3) ⑤
The matrix $M$ has eigenvalue 12
repeating 4 times.

Choose $\lambda = 12$
Calculate $M - \lambda I$; we get a matrix.

If $MV_1 = \lambda V_1$

$MV_2 = \lambda V_2 + V_1$

we get $(M - \lambda I)V_1 = 0 \Rightarrow (M - \lambda I)^2 V_2 = 0$

$(M - \lambda I)V_2 = V_1$

Choose $V_2 : (M - \lambda I)V_2 \neq 0, (M - \lambda I)^2 V_2 = 0$

One choice is $V_2 = \begin{pmatrix} 1 \\ 10 \\ 0 \end{pmatrix}$; $V_1 = (M - \lambda I)V_2 = \begin{pmatrix} 33 \\ -60 \\ 45 \\ -12 \end{pmatrix}$
\[ V_1 = \begin{pmatrix} 3 \newline -6 \newline 45 \newline -12 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 \newline 0 \newline 0 \end{pmatrix} \]
is a chain of gen. eigenvectors.

\[ V_1 = (M - 12 \mathbb{I}) \cdot V_2 \]
\[ (M - 12 \mathbb{I})^2 \cdot V_2 = 0 \]

Likewise

\[ V_3 = \begin{pmatrix} 315 \newline -756 \newline 639 \newline -180 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0 \newline 0 \newline 1 \end{pmatrix} \]
is another chain of gen. eigenvectors.

\[ \Theta \]
(5) In this problem please do not manufacture a diagonal matrix. I am looking for a genuine (i.e. non diagonal) matrix. One way to do this is the following:

- Let $A$ be any symmetric matrix. (Don't choose $A$ to be diagonal).
- Eigenvalues of $A$ are real, but could be negative. Write $B = \exp(A)$ ← Use MATLAB here.
- If $\lambda$ is an eigenvalue of $A$
  - $e^\lambda$ is an eigenvalue of $B$.
  - Hence eigenvalues of $B$ are all positive.
- $B^T = (e^A)^T = e^{AT} = e^A = B$
  - Hence, $B$ is also symmetric.
B is the required answer.

- Eigenvectors of B are all l.i. and orthogonal to each other (cf. property of symmetric p.d. matrices)
- Let P be the matrix of eigenvectors of B. Then
  \[ P^T = P^{-1} \quad \text{because} \quad PP^T = I \]
  \[ \text{property of orthogonal eigenvectors} \]
- \[ P^{-1}BP = P^TBP \] is a diagonal matrix of eigenvalues of B.
Hint for Problem 6:

Assume that the matrix $A$ has eigenvalues at $\lambda_i$ with eigenvectors at $v_i$, for $i = 1, \ldots, n$. If the eigenvectors $v_1, v_2, \ldots, v_n$ are linearly dependent it would follow that either $v_1, v_2, \ldots, v_{n-1}$ are linearly dependent or $v_1, v_2, \ldots, v_{n-1}$ are linearly independent and

$$v_n = \alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1}.$$ 

It would follow that

$$Av_n = \alpha_1 Av_1 + \cdots + \alpha_{n-1} Av_{n-1},$$

which implies that

$$\lambda_n v_n = \alpha_1 \lambda_1 v_1 + \cdots + \alpha_{n-1} \lambda_{n-1} v_{n-1},$$

i.e.

$$v_n = \alpha_1 \frac{\lambda_1}{\lambda_n} v_1 + \cdots + \alpha_{n-1} \frac{\lambda_{n-1}}{\lambda_n} v_{n-1}.$$ 

Since $v_1, \ldots, v_{n-1}$ are linearly independent it would follow that

$$\alpha_i = \alpha_1 \frac{\lambda_i}{\lambda_n},$$

for $i = 1, \ldots, n - 1$. Since the eigenvalues are all distinct, this would imply that $\alpha_i = 0$ for all $i = 1, \ldots, n - 1$. This would imply that $v_n$ must be the zero vector which violates the assumption that $v_n$ is an eigenvector and must necessarily be nonzero.

Hint for Problem 7:

1) We have $AP = PB$. Also $Bv = \lambda v$ which implies that $PBv = P\lambda v = \lambda P v$. Hence it follows that $A(Pv) = \lambda(Pv)$.

2) From Cayley Hamilton Theorem we have

$$A^3 - c_3 A^2 - c_2 A - c_1 I = 0.$$ 

Need to show that

$$A[v_1, v_2, v_3] = [v_1, v_2, v_3] \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$ 

This would imply

$$Av_1 = c_1 v_3, \quad Av_2 = v_1 + c_2 v_3, \quad Av_3 = v_2 + c_3 v_3.$$ 

The idea is to show the above three relations using the definition of $v_1, v_2$ and $v_3$ and the Cayley Hamilton Theorem. (Good Luck)

3) You need to define $v_1, v_2, v_3, v_4, v_5$ from the pattern in part 2 of the problem and then verify that it is correct.

Hint for Problem 8:

When matrices commute, you can use the binomial formula. Because one of the matrix is nilpotent, the binomial expansion stops after finitely many terms. Hope this helps.