Why study vectors

A. In physics to talk about position, velocity, and acceleration of a point object.

Ex: 1

\[ \mathbf{r}(t) = \begin{pmatrix} \sin t \\ \cos t \\ t \end{pmatrix} \]

be the position of a particle at time instant \( t \).
\[ V(t) = \dot{r}(t) \leq \text{Velocity} \]
\[ a(t) = \ddot{r}(t) \leq \text{acceleration} \]

\[ v(t) = \begin{pmatrix} \cos t \\ -\sin t \\ 1 \end{pmatrix} \]

\[ \text{speed } \dot{s}(t) = \| v(t) \| = \sqrt{2} \]

\[ a(t) = \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix} \]

The velocity vector is always pointed tangential to the curve shown in Fig.
velocity vector is shown tangent to the curve.

Define \( u(t) = \frac{v(t)}{||v(t)||} \)

\( u(t) \) is the unit velocity vector or the unit tangent vector.

\[
||v(t)|| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}
\]

\[
u(t) = \left( \frac{\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)
\]
The acc\textsuperscript{n} vector \( a(t) \) is perpendicular to \( u(t) \). It is called the normal acc\textsuperscript{n}.

\[
q_N(t) = \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix}
\]

The tangential component of the acc\textsuperscript{n} vector is zero.

Remark: This is not always the case but only for this problem.
\[ \mathbf{r}(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \] 
position vector

\[ \mathbf{v}(t) = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} \] 
velocity vector

\[ ||\mathbf{v}(t)|| = \sqrt{1 + 4t^2 + 9t^4} \]

\[ \mathbf{u}(t) = \begin{pmatrix} \left(1 + 4t^2 + 9t^4\right)^{-\frac{1}{2}} \\ 2t \left(1 + 4t^2 + 9t^4\right)^{-\frac{1}{2}} \\ 3t^2 \left(1 + 4t^2 + 9t^4\right)^{-\frac{1}{2}} \end{pmatrix} \] 
unit tangent vector.
In general, the acc vector does not point in the tangential direction.

\[ a_N(t) = a(t) - a_T(t) \text{ Normal component of the acc} \]

\[ a_T(t) = \text{proj}_{[u(t)]} a(t) \text{ Tangential component of the acc} \]
\[ a(t) = \begin{pmatrix} 0 \\ 2 \\ 6t \end{pmatrix} \]

Define
\[ a_T(t) = \text{proj}_{V(t)} \mathbf{a}(t) \]
\[ a_T(t) = \alpha V(t) \]
\[ [a - \alpha V(t)] \cdot [V(t)] = 0 \]
\[ \alpha = \frac{a \cdot V}{V \cdot V} \]
\[ a_T(t) = \frac{a \cdot V}{V \cdot V} V \]
\[ a \cdot V = 4t + 18t^3 \]
\[ V \cdot V = 1 + 4t^2 + 9t^4 \]
\[ a_T(t) = \frac{2t(2 + 9t^2)}{1 + 4t^2 + 9t^4} \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} \]
\[ a_N(t) = a - a_T(t) \]

\[
= \left( -\frac{2t(2+5t^2)}{1+4t^2+9t^4} \right) \]

\[
= \left( 2 - \frac{4t^2(2+5t^2)}{1+4t^2+9t^4} \right) \]

\[
= \left( 6t - \frac{6t^3(2+5t^2)}{1+4t^2+9t^4} \right) \]
Unit normal vector

\[ \mathbf{u}(t) = \frac{\mathbf{a}_N(t)}{\|\mathbf{a}_N(t)\|} \]

\[ = \begin{pmatrix} -t(2+9t^2) \\ 1-9t^4 \\ 3t(1+2t^2) \end{pmatrix} \]

\[ = \sqrt{1 + 13t^2 + 54t^4 + 117t^6 + 81t^8} \]
Example 1 (continued)

Note that

\[ U(t) = \begin{pmatrix} \cos t / \sqrt{2} \\ -\sin t / \sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad \text{unit tangent vector} \]

\[ P(t) = \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix} \quad \text{unit normal vector}. \]

If we compute \( \dot{U}(t) \) we obtain

\[ \dot{U}(t) = \frac{1}{\sqrt{2}} P(t) \]

Thus \( \dot{U}(t) \) is in the same direction as \( P(t) \). \( \frac{1}{\sqrt{2}} / \sqrt{2} \) is the curvature \( = \frac{1}{2} \).
In general we write

\[ \dot{\alpha}(t) = \kappa(t) \rho(t) \]

\[ \frac{\kappa}{\rho} \]

Curvature

In this case we have

\[ \kappa(t) = \frac{1}{\sqrt{2}} \]

Curvature = \frac{1}{2}

Remark: A straight line has zero curvature. A circle has constant curvature.

"Curvature" together with another quantity called "torsion" measures the shape of a curve in \( \mathbb{R}^3 \).
B Vectors are useful in calculating directional derivatives.

Consider the top hemisphere of a sphere of radius $R$ centered at $(0, 0, 0)$. The height of the sphere at any point $(x, y)$ is given by...
\[ h(x, y) = \sqrt{R^2 - x^2 - y^2} \]

(See Figure)

We are interested in the rate at which the height changes along any direction \( \mathbf{V}_a \) starting at \( \mathbf{a} = (x_0, y_0) \).
In co-ordinates let us write

\[ V_a = \begin{pmatrix} \xi a \\ \eta a \end{pmatrix} \] (See Figure)

Note that in order to write this we need to translate \( V_a \) to the origin.

Def: The directional derivative of the function \( h(x, y) \) in the direction \( V_a \) at the point \( a \) is given by

\[ V_a h(x_0, y_0) = \xi a \frac{\partial h(x_0, y_0)}{\partial x} + \eta a \frac{\partial h(x_0, y_0)}{\partial y} \]
Note that
\[
\frac{\partial h}{\partial x} \bigg|_{x=x_0, \ y=y_0} = -\frac{x_0}{\sqrt{R^2 - x_0^2 - y_0^2}}
\]
\[
\frac{\partial h}{\partial y} \bigg|_{x=x_0, \ y=y_0} = -\frac{y_0}{\sqrt{R^2 - x_0^2 - y_0^2}}
\]

\[V_a h (x_0, y_0) = -\frac{\xi_a x_0 + \eta_a y_0}{\sqrt{R^2 - x_0^2 - y_0^2}}\]
The directional derivative measures the rate at which a function $h(x,y)$ changes at a pt. '$A$' in a given direction $V_a$.

Many often, one choose $V_a$ to be an unit vector

$$V_a = \begin{pmatrix} \xi_a \\ \eta_a \end{pmatrix} \quad \xi_a^2 + \eta_a^2 = 1$$

describing only direction.
The vector

\[
\begin{pmatrix}
\frac{\partial h}{\partial x}(x_0, y_0) \\
\frac{\partial h}{\partial y}(x_0, y_0)
\end{pmatrix}
\]

is called the gradient of \( h \) at the point \( (x_0, y_0) \) and is denoted by \( \nabla h(x_0, y_0) \).

Using this notation we can write

\[
\nabla_a h(x_0, y_0) = \nabla h(x_0, y_0) \cdot \nabla a
\]

which is the dot product of the gradient vector and the direction vector.
Directional derivative can be positive or negative.

+ve indicates that the function is increasing along the direction $\mathbf{V}_a$.

-ve indicates that the function is decreasing along the direction $\mathbf{V}_a$.

**Q:** Assume $\mathbf{V}_a$ is an unit vector. Find $\mathbf{V}_a$ such that $V_{\mathbf{a}} h(x_0, y_0)$ is maximum.

**A:** 

$$V_{\mathbf{a}} h(x_0, y_0) = \left\| \nabla h(x_0, y_0) \right\| \frac{1}{\left\| \mathbf{V}_a \right\|} \cos \alpha$$

$$= \left\| \nabla h(x_0, y_0) \right\| \cos \alpha$$

For $V_{\mathbf{a}} h(x_0, y_0)$ to be maximum we
have \( \theta = 0, \cos \theta = 1 \)

Hence

\[
\nabla q = \frac{\nabla h(x_0, y_0)}{\| \nabla h(x_0, y_0) \|}
\]

For our problem

\[
\nabla h(x_0, y_0) = -\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \sqrt{R^2 - x_0^2 - y_0^2}
\]

\[
\frac{\nabla h(x_0, y_0)}{\| h(x_0, y_0) \|} = -\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \sqrt{\frac{x_0^2 + y_0^2}{R^2 - x_0^2 - y_0^2}}
\]
Direction along which the height increases most rapidly.
C. Vectors are useful in describing tangent planes to a surface.

Let \( \mathbf{a} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \) be a pt. on the top hemisphere of a sphere:

\[ x^2 + y^2 + z^2 = R^2 \]

Define \( \phi(x, y, z) = x^2 + y^2 + z^2 - R^2 \)
The sphere is given by

$$\phi(x, y, z) = 0.$$ 

Our goal is to approximate the sphere up to a tangent plane at point \(a\). What is the equation of the tangent plane?

If we write

$$\phi(x, y, z) = \phi(x_0, y_0, z_0)$$

$$+ \frac{\partial \phi}{\partial x} \bigg|_{x} (x-x_0)$$

$$+ \frac{\partial \phi}{\partial y} \bigg|_{y} (y-y_0)$$

$$+ \frac{\partial \phi}{\partial z} \bigg|_{z} (z-z_0) + \text{h.o.t.}$$
it follows that up to terms linear in \( x, y \) & \( z \) we write

\[
\phi (x, y, z) \sim \nabla \phi (x_0, y_0, z_0) \cdot \begin{pmatrix}
    x - x_0 \\
    y - y_0 \\
    z - z_0
\end{pmatrix}
\]

where

\( \nabla \phi \) is the famous gradient vector

\[
\nabla \phi = \begin{pmatrix}
    \frac{\partial \phi}{\partial x} \\
    \frac{\partial \phi}{\partial y} \\
    \frac{\partial \phi}{\partial z}
\end{pmatrix}
\]
For our problem,

\[ \nabla \Phi(x_0, y_0, z_0) = \begin{pmatrix} 2x_0 \\ 2y_0 \\ 2z_0 \end{pmatrix} \]

Hence,

\[ \Phi(x, y, z) \approx 2x_0(x - x_0) + 2y_0(y - y_0) + 2z_0(z - z_0) \]

Tangent plane is given by

\[ 2x_0(x - x_0) + 2y_0(y - y_0) + 2z_0(z - z_0) = 0 \]

\[ \Rightarrow x_0x + y_0y + z_0z = x_0^2 + y_0^2 + z_0^2 \]

Eqn of the tangent plane = \( R^2 \)
Note that the gradient vector is a vector pointing outwards and is perpendicular to the tangent plane.

Gradient vector is an outward normal vector.
Example:

\[ z \]

Our last example is an example of "magic carpet" given by the equation:

\[ z = S(x, y). \]

Here

\[ \phi(x, y, z) = z - S(x, y) \]

\[ \nabla \phi = \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{pmatrix} = \begin{pmatrix} -\frac{\partial S}{\partial x} \\ -\frac{\partial S}{\partial y} \\ 1 \end{pmatrix} \]
At any pt. \((x_0, y_0, s(x_0, y_0))\) on the carpet,

\[
\nabla \Phi(x) = \begin{pmatrix}
-\frac{\partial s}{\partial x}(x_0, y_0) \\
-\frac{\partial s}{\partial y}(x_0, y_0)
\end{pmatrix}
\]

Tangent plane is given by

\[
-\frac{\partial s}{\partial x}(x_0, y_0)(x-x_0) - \frac{\partial s}{\partial y}(x_0, y_0)(y-y_0) + 1(z-z_0) = 0
\]

\[
\Rightarrow (z-z_0) = \frac{\partial s}{\partial x}(x_0, y_0)(x-x_0) + \frac{\partial s}{\partial x}(x_0, y_0)(y-y_0)
\]